Dp-finite Hahn Series Fields

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1 Introduction

One of the tasks model theory has assumed is the classification of first-order mathematical structures. Taking Shelah's work as a starting point, it becomes evident that the model-theoretic properties that allow us to draw dividing lines between desirable theories – regarding tameness of definable sets, or the number of non-isomorphic models of a given cardinality – rely fundamentally on "coding" combinatorial properties.

In particular, a theory is *dependent* (also known as NIP) if it is not possible to codify the relation $\{(i, S) \subseteq \omega \times \mathcal{P}(\omega) : i \in S\}$. More precisely, an $\mathscr{L}$-theory T is dependent if it is not possible to find some $\mathscr{L}$-formula $\varphi(x; y)$ and some sequences $(a_i : i < \omega), (b_S : S \subseteq \omega)$ of tuples from a fixed monster model of T such that $\models \varphi(a_i; b_S)$ if and only if $i \in S$. Once the combinatorial property is fixed, it is common to define some notion of rank or dimension in order to measure the complexity of types and definable sets, such as Morley rank in ω-stable theories, D-rank in simple theories or SU-rank in supersimple theories. One suitable rank notion for types in dependent theories T is the *dp-rank*. Dependent theories are exactly those in which dp-rank is always bounded by $|T|^+$.

Additionally, Shelah-style model-theoretic hypotheses on algebraic structures have proved to have useful consequences for the structures themselves. For example, it is well known that any infinite ω-stable field is algebraically closed [14], and that a dp-minimal – i.e. of dp-rank 1 – field is either separably closed, real closed or admits a non-trivial definable henselian valuation [10].

Last year, Christian d'Elbée and Yatir Halevi [3] published a paper on algebraic properties of dp-minimal integral domains. Namely, they proved that any such ring is local, and that in order for it to be a valuation ring, it is necessary and sufficient that its residue field is either infinite or, if it is finite, that its maximal ideal is principal.

With this in mind, the present document consists of exploring some examples of dp-finite fields and domains. First attempts to classify dp-finite fields were motivated by a conjecture of Shelah: any NIP field is either separably closed, real closed or admits a non-trivial definable henselian valuation. Johnson [11] proved this conjecture for dp-finite fields, and gave an algebraic classification thereof. Using results of Dolich and Goodrick [4] on the classification of dp-finite ordered abelian groups (proved independently by Farré [6] and Halevi-Hasson [7]), we will focus on the class of *Hahn series* fields, which seem to be natural examples of structures to study within the context of dp-finite algebraic structures: they form a natural class that "glues" together ordered abelian groups and fields.

This work is organized as follows. In Section 2.1 we will introduce the definition of dp-rank and explore some Shelah-style arrays that allow us to compute it. In Section 2.2 we fix some notation around valued fields and valuation rings, which will be used throughout the document. In Section 2.3 we present some important results on henselianity of valued fields. This section splits into two subsections, one in which we define our structure of interest and prove some theorems around a well-behaved "weak convergence" – which, in the literature, is known as Kaplansky Theory – and another in which we introduce the reader to the model theory of valued fields by means of a crucial theorem on relative quantifier elimination due to Pas.

In Section 3.1 we state the main classification theorems of dp-finite fields and dp-finite ordered abelian groups. Using a result of Sinclair [17], we determine which Hahn series fields are dp-finite and explore the dp-ranks of such structures computed in some reducts of the language. Finally, in Section 3.2, we study the structure of ideals of some valuation rings, motivated by some results of d'Elbée and Halevi [3].

2 Preliminaries

2.1 Dp-ranks

In this section we will define dp-rank by means of mutually indiscernible sequences and show some equivalent definitions. The proofs will mostly be omitted, but the reader can find them in chapter 4 of [16], a reference on which this section is based.

Let T be a theory with infinite models.

Definition 2.1.1. An $\mathscr{L}$-theory T is *dependent*, also known as *NIP*, if there is no $\mathscr{L}$-formula $\varphi(x; y)$ and no sequences $(a_i : i < \omega), (b_S : S \subseteq \omega)$ of tuples from a model of T such that $\models \varphi(a_i; b_S)$ if and only if $i \in S$.

We may work in a monster model $\mathfrak{C}$ of T, that is, a model which is κ-saturated for some $\kappa > |T|$. All elements and sets of parameters are assumed to come from this monster model. Let us start with a proposition about mutually indiscernible sequences in $\mathfrak{C}$ under the hypothesis of dependence.

Fact 2.1.2 (Cf. Proposition 4.8 of [16])**.** *Let T be a dependent theory, and let $(I_t : t \in X)$ be a family of sequences mutually indiscernible over some set A. Let b be a finite tuple. Then there is a set $X_b \subseteq X$ of size at most $|T|$ such that the sequences $(I_t : t \in X \setminus X_b)$ are mutually indiscernible over Ab.*

Let π be a partial type over a set A and let κ be a cardinal, finite or infinite. We say that dp-rk$(\pi, A) < \kappa$ if for every family $(I_t : t < \kappa)$ of mutually indiscernible sequences over A and every $b \models \pi$, there is some $t < \kappa$ such that I_t is indiscernible over Ab. If $b \in \mathfrak{C}$, then dp-rk(b, A) is defined to be dp-rk$(tp(b/A), A)$. We say that dp-rk$(\pi, A) = \kappa$ if dp-rk$(\pi, A) < \kappa^+$ but it is not the case that dp-rk$(\pi, A) < \kappa$.

Proposition 2.1.3. *The following propositions are equivalent:*

1. *T is NIP.*
2. *For any finitary type p and any set A,* dp-rk$(p, A) < |T|^+$.
3. *There is some infinite cardinal κ such that for any finitary type p and any set A,* dp-rk$(p, A) < \kappa$.

Proof. 1) $\Longrightarrow$ 2). Let $(I_t : t < |T|^+)$ be a family of mutually indiscernible sequences over A, and let $b \models p$. Use Fact 2.1.2 to find a subset X of $|T|^+$ of size at most $|T|$ such that $(I_t : t \in |T|^+ \setminus X)$ is a family of mutually indiscernible sequences over Ab. Since $|T|^+ \setminus X$ is of size $|T|^+$, the result follows.

2) $\Longrightarrow$ 3). Choose $\kappa = |T|^+$.

3) $\Longrightarrow$ 1). Suppose $\varphi(x, y)$ has IP, and let κ be any infinite cardinal. By compactness, if $X = \omega \times \kappa$ is ordered lexicographically, there is an indiscernible sequence $(a_i : i \in X)$ and a tuple b such that $\varphi(a_i, b)$ holds if and only if $i = (0, \alpha)$ for some $\alpha < \kappa$. The family of sequences $(I_t : t < \kappa)$ given by $I_t = (a_{(n,t)} : n < \omega)$ is mutually indiscernible, but none is indiscernible over b because $\models \varphi(a_{(0,t)}, b) \wedge \neg\varphi(a_{(1,t)}, b)$ for any $t < \kappa$. Therefore dp-rk$(b, \emptyset) \geq \kappa$.

$\square$

Lemma 2.1.4. *Let π be a partial type over A and κ be any cardinal. If $A \subseteq B$, then* dp-rk$(\pi, A) < \kappa$ *if and only if* dp-rk$(\pi, B) < \kappa$.

Proof. Suppose that dp-rk$(\pi, A) < \kappa$ and let $(I_t : t < \kappa)$ be a family of mutually indiscernible sequences over B. Let $\overline{b}$ enumerate B and, if $I_t = (a_i^t : i \in \mathcal{I}_t)$, where $\mathcal{I}_t$ is the index set of I_t, let $I'_t = (a_i^t{}^\frown \overline{b} : i \in \mathcal{I}_t)$. It follows that the family $(I'_t : t < \kappa)$ is mutually indiscernible over A. If $b \models \pi$, then there is some $t < \kappa$ such that I'_t is indiscernible over Ab, but then I_t is indiscernible over $A\overline{b}b = Bb$.

Now suppose that dp-rk$(\pi, B) < \kappa$ and let $(I_t : t < \kappa)$ be a family of mutually indiscernible sequences over A. Suppose there is some $b \models \pi$ such that none of the I_t is indiscernible over Ab. For any $t < \kappa$, let $I'_t \models EM(I_t/BI'_{<t}I_{>t})$ be an indiscernible sequence over $BI'_{<t}I_{>t}$. Then $(I'_t : t < \kappa)$ is a family of mutually indiscernible sequences over B, and, in particular, $\mathrm{tp}((I_t)_{t<\kappa}/A) = \mathrm{tp}((I'_t)_{t<\kappa}/A)$. Therefore none of the I'_t is indiscernible over Ab' for some $b' \equiv_A b$, contradicting the hypothesis. □

The next theorem summarizes how to bound dp-rank by means of mutually indiscernible sequences.

Theorem 2.1.5. *Let π be a partial type over A and let κ be any cardinal. The following propositions are equivalent.*

1. dp-rk$(\pi, A) < \kappa$,
2. *For any family $(I_t : t \in X)$ of sequences, mutually indiscernible over A, and any $b \models \pi$, there is a subset $X_b \subseteq X$ of size less than κ such that $(I_t : t \in X \setminus X_b)$ are mutually indiscernible over Ab.*
3. *For any family $(I_t : t \in X)$ of sequences, mutually indiscernible over A, and any $b \models \pi$, there is a subset $X_b \subseteq X$ of size less than κ such that I_t is indiscernible over Ab for any $t \in X \setminus X_b$.*
4. *For any family $(I_t : t \in X)$ of sequences, mutually indiscernible over A, and any $b \models \pi$, there is a subset $X_b \subseteq X$ of size less than κ such that all the elements of I_t have the same type over Ab for any $t \in X \setminus X_b$.*

Proof. 1) $\Longrightarrow$ 2). This is Proposition 4.17 of [16].

2) $\Longrightarrow$ 3). For any $t \in X \setminus X_b$, I_t is indiscernible over $(\bigcup_{t \in X \setminus (X_b \cup \{t\})} I_t) \cup Ab$, therefore indiscernible over Ab.

3) $\Longrightarrow$ 4). Let a_i, a_j be elements of I_t with $i < j$ and $t \in X \setminus X_b$. By indiscernibility over Ab, $\models \varphi(a_i, \overline{a}, b) \longleftrightarrow \varphi(a_j, \overline{a}, b)$ for any formula $\varphi(x, y, z)$ and any tuple $\overline{a}$ of A.

4) $\Longrightarrow$ 1). Suppose otherwise, so that there is some family $(I_t : t < \kappa)$ of mutually indiscernible sequences over A and some $b \models \pi$ such that none of the I_t is indiscernible over Ab. Then, for each t, there is a formula $\varphi(x, y, z)$, a tuple $\overline{a}$ of A, and two tuples $(a_{i_1}, \ldots, a_{i_n}), (a_{j_1}, \ldots, a_{j_n})$ of I_t with $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$, such that, say, $\models \varphi(a_{i_1}, \ldots, a_{i_n}, \overline{a}, b) \wedge \neg\varphi(a_{j_1}, \ldots, a_{j_n}, \overline{a}, b)$. Therefore there is some $k_t \in \{1, \ldots, n\}$ such that

$$\models \varphi(a_{i_1}, \ldots, a_{i_{k_t-1}}, a_{i_{k_t}}, a_{j_{k_t+1}}, \ldots, a_{j_n}, \overline{a}, b) \wedge \neg\varphi(a_{i_1}, \ldots, a_{i_{k_t-1}}, a_{j_{k_t}}, a_{j_{k_t+1}}, \ldots, a_{j_n}, \overline{a}, b),$$

so that $\mathrm{tp}(a_{i_{k_t}}/AbC_t) \neq \mathrm{tp}(a_{j_{k_t}}/AbC_t)$ if $C_t = \{a_{i_1}, \ldots, a_{i_{k_t-1}}, a_{j_{k_t+1}}, \ldots, a_{j_n}\}$. Without loss, $i_{k_t} < j_{k_t}$, so if the order type of I_t were dense, the family $(I'_t : t < \kappa)$ given by $I'_t = \{a_i{}^\frown C_t \in I_t : i_{k_t} \leq i \leq j_{k_t}\}$ would be a family of mutually indiscernible sequences over A in which any sequence has a couple of elements with different type over Ab. This order type can be achieved by using compactness at the very first stage of the proof so that, for instance, any I_t has order type $(\mathbb{Q}, <)$.

□

Corollary 2.1.6. *Let $a, b \in \mathfrak{C}$, $A \subset \mathfrak{C}$, and let κ_1, κ_2 be cardinals such that* dp-rk$(b/A) < \kappa_1$ *and* dp-rk$(a/Ab) < \kappa_2$. *Then* dp-rk$(a, b/A) + 1 < \kappa_1 + \kappa_2$.

Proof. Let $(I_t : t \in X)$ be a family of mutually indiscernible sequences over A with $|X| = \kappa_1 + \kappa_2 - 1$ and let $a', b' \equiv_A a, b$. Then $b' \equiv_A b$, so there is some set $X_{b'} \subseteq X$ of size less than κ_1 such that $(I_t : t \in X \setminus X_{b'})$ is a family of mutually indiscernible sequences over Ab'. Since dp-rk$(a'/Ab') < \kappa_2$, there is some set $X_{a'} \subseteq X \setminus X_{b'}$ of size less than κ_2 such that $(I_t : t \in X \setminus X_{a'}X_{b'})$ is a family of mutually indiscernible sequences over $Aa'b'$. □

We would like to compute dp-ranks even if we do not have a full characterization of indiscernible sequences. To this end, we need the following

Definition 2.1.7. Let $\pi(x)$ be a partial type over a set A. We define $\kappa_{ict}(\pi, A)$ as the minimal κ such that the following does not exist:

1. formulas $\varphi_\alpha(x; y_\alpha)$ indexed by κ,
2. an array $(a_i^\alpha : (i, \alpha) \in \omega \times \kappa)$ of tuples such that $|a_i^\alpha| = |y_\alpha|$,
3. for every $\eta : \kappa \longrightarrow \omega$, a tuple $b_\eta \models \pi$ such that
$$\models \varphi_\alpha(b_\eta; a_i^\alpha) \Longleftrightarrow \eta(\alpha) = i.$$

Such a family of tuples and formulas is called an *ict-pattern for π of length κ (and width ω).*

Proposition 2.1.8. *Let T be a dependent theory. For any partial type π over A and any cardinal κ,* dp-rk$(\pi, A) < \kappa$ *if and only if $\kappa_{ict}(\pi, A) \leq \kappa$*

Proof. Suppose $\kappa_{ict}(\pi, A) > \kappa$, and fix $\lambda = \beth_{\left(2^{|T|+|A|}\right)^+}$. By compactness, there is an ict-pattern of length κ and width λ as in the definition. Define $I_\alpha = (a_i^\lambda : i < \lambda)$ for any $\alpha < \kappa$. Then there is an A-indiscernible sequence $I'_\alpha = (b_i^\alpha : i < \omega)$ and a pair of indices $i_0^\alpha < i_1^\alpha < \lambda$ such that
$$b_0^\alpha b_1^\alpha \equiv_A a_{i_0^\alpha}^\alpha a_{i_1^\alpha}^\alpha. \tag{2.1.1}$$
Inductively define the sequences $I''_\alpha = (c_i^\alpha : i < \omega)$ satisfying $I''_\alpha \models EM(I'_\alpha / A I''_{<\alpha} I'_{>\alpha})$ and being indiscernible over $A I''_{<\alpha} I'_{>\alpha}$. This implies that the family $(I''_\alpha : \alpha < \kappa)$ is mutually indiscernible over A, and that
$$I''_\alpha \equiv_A I'_\alpha \tag{2.1.2}$$
by A-indiscernibility of the I'_α for any $\alpha < \kappa$. Let $\eta : \kappa \longrightarrow \lambda$ be given by $\eta(\alpha) = i_0^\alpha$, so that there is some $b \models \pi$ such that for any $\alpha < \kappa$ we have that $a_{i_0^\alpha} \not\equiv_{Ab} a_{i_1^\alpha}$. Conjugating over A two times, according to equations 2.1.1 and 2.1.2, there is some $b' \models \pi$ such that $c_0^\alpha \not\equiv_{Ab'} c_1^\alpha$ for any $\alpha < \kappa$, which contradicts dp-rk$(\pi, A) < \kappa$ for it implies that there is at least some $\alpha < \kappa$ such that all the elements of I''_α have the same type over Ab'.

Now suppose that dp-rk$(\pi, A) \geq \kappa$, so that there is a family $(I_\alpha : \alpha < \kappa)$ of mutually indiscernible sequences over A, say, with $I_\alpha = (a_i^\alpha : i < \omega)$, a realization $b \models \pi$ and a set of formulas $\{\varphi_\alpha(x_\alpha, y) : \alpha < \kappa\}$ such that for any $\alpha < \kappa$ we have that $\models \varphi_\alpha(b; a_0^\alpha) \wedge \neg\varphi_\alpha(b; a_1^\alpha)$. Let $\psi_\alpha(x; y_\alpha, y'_\alpha) = \varphi_\alpha(x; y_\alpha) \wedge \neg\varphi_\alpha(x; y'_\alpha)$ and let $J_\alpha = (a_{2i}^\alpha {}^\frown a_{2i+1}^\alpha : i < \omega)$. Then there are finitely many elements of J_α that satisfy $\psi_\alpha(b; y_\alpha, y'_\alpha)$, because otherwise the truth value of $\varphi_\alpha(b; a_i^\alpha)$ would alternate infinitely many times, in contradiction with NIP.

If we remove such pairs of the original sequences, we may assume that $a_0^\alpha {}^\frown a_1^\alpha$ is the only pair that satisfies $\psi_\alpha(b; y_\alpha, y'_\alpha)$. Then, by mutual indiscernibility of the original sequences, for any finite set of ordinals $\{\alpha_1, \ldots, \alpha_n\}$ and after conjugating n times the element b over A, for any $\eta \in {}^{\{\alpha_1,\ldots,\alpha_n\}}\omega$ there is some $b_\eta \models \pi$ such that $\models \bigwedge_{j=1}^{n} \psi_{\alpha_j}(b_\eta; a_{2i}^{\alpha_j}, a_{2i+1}^{\alpha_j})$ exactly when $\eta(\alpha_j) = i$. Hence $\kappa_{ict}(\pi, A) > \kappa$ follows by compactness. □

The following definition and facts can be found in [1].

Definition 2.1.9. Let κ be a cardinal. We say that T admits an *inp-pattern of length* κ if there is a sequence of formulas $\varphi_\alpha(x; y_\alpha)$ indexed by κ, with x a single free variable, and an array $(a_i^\alpha : (i, \alpha) \in \omega \times \kappa)$ in a model $\mathfrak{M}$ of T such that $|a_i^\alpha| = |y_\alpha|$, and satisfying

1. for any $\alpha < \kappa$, there is some $k_\alpha < \omega$ such that the set $\{\varphi_\alpha(x; a_i^\alpha) : i < \omega\}$ is k_α-inconsistent,
2. for any $\eta : \kappa \longrightarrow \omega$, the set $\{\varphi_\alpha(x; a_{\eta(\alpha)}^\alpha) : \alpha < \kappa\}$ is consistent.

The least cardinal κ for which there is no inp-pattern of length κ is called κ_{inp}, and the supremum of all lengths of inp-patterns is called the *burden* of T. Therefore the burden of T is less than κ if and only if $\kappa_{inp} \leq \kappa$ for any cardinal κ.

Fact 2.1.10 (Cf. Proposition 6 of [1])**.** *For any inp-pattern (resp. ict-pattern) of length κ, there is some other inp-pattern (resp. ict-pattern) of the same length, such that the parameters of the rows form a mutually indiscernible family.*

Fact 2.1.11 (Cf. Proposition 10 of [1])**.** *Let T be a dependent theory and let x be a single variable. Then $\kappa_{inp} = \kappa_{ict}(x = x, \emptyset)$.*

Proof. Let κ be any cardinal and suppose $\{\varphi_\alpha(x; a_i^\alpha) : i < \omega, \alpha < \kappa\}$ is an inp-pattern of length κ. We know that the set $\{\varphi_\alpha(x; a_0^\alpha) : \alpha < \kappa\}$ is consistent, say, witnessed by b. Then, for each $\alpha < \kappa$ there are finitely many a_i^α with $i > 0$ satisfying $\varphi_\alpha(b; a_i^\alpha)$, so after taking them out of the rows, we may assume that $\varphi_\alpha(b; a_i^\alpha)$ holds if and only if $i = 0$. If the family of sequences $(I_\alpha : \alpha < \kappa)$ given by $I_\alpha = (a_i^\alpha : i < \omega)$ is mutually indiscernible, then for any finite set of ordinals $\{\alpha_1, \ldots, \alpha_n\}$ and after conjugating n times the element b, for any $\eta \in {}^{\{\alpha_1,\ldots,\alpha_n\}}\omega$ there is some b_η such that $\models \bigwedge_{j=1}^{n} \varphi_{\alpha_j}(b_\eta; a_{\eta(\alpha_j)}^{\alpha_j}) \wedge \neg\varphi_{\alpha_j}(b_\eta; a_i^{\alpha_j})$ for any $i \neq \eta(\alpha_j)$. Hence $\kappa_{ict}(x = x, \emptyset) > \kappa$ follows by compactness. We may assume indeed that $(I_\alpha : \alpha < \kappa)$ is a mutually indiscernible family following the "extend-and-extract" argument of the proof of Fact 2.1.10. Therefore $\kappa_{inp} \leq \kappa_{ict}(x = x, \emptyset)$.

Let κ be any cardinal and suppose $\{\varphi_\alpha(x; a_i^\alpha) : i < \omega, \alpha < \kappa\}$ is an ict-pattern of length κ. Again, by extending and extracting – i.e. Fact 2.1.10 – we may assume that the parameters form a mutually indiscernible family of rows. We claim that, for any $\alpha < \kappa$, the set $\{\varphi_\alpha(x; a_{2i}^\alpha) \wedge \neg\varphi_\alpha(x; a_{2i+1}^\alpha) : i < \omega\}$ is k_α-inconsistent for some $k_\alpha < \omega$. If this is not the case, then we would be able to find arbitrarily many indices $i < \omega$ in which $\models \varphi_\alpha(b; a_{2i}^\alpha) \wedge \neg\varphi_\alpha(b; a_{2i+1}^\alpha)$ for some common b, so φ_α itself would not have finite alternation rank, contradicting NIP. If $\eta \in {}^\kappa\omega$, then by the ict-condition, there is some $b_{2\eta}$ such that for any $\alpha < \kappa$ we have that $\models \varphi_\alpha(b_{2\eta}, a_{2\eta(\alpha)}^\alpha)$ and $\not\models \varphi_\alpha(b_{2\eta}, a_i^\alpha)$ for any $i \neq 2\eta(\alpha)$, so, in particular, $b_{2\eta}$ satisfies the set $\{\varphi_\alpha(x; a_{2\eta(\alpha)}^\alpha) \wedge \neg\varphi_\alpha(x; a_{2\eta(\alpha)+1}^\alpha) : \alpha < \kappa\}$. To sum up, the set

$$\{\varphi_\alpha(x; a_{2i}^\alpha) \wedge \neg\varphi_\alpha(x; a_{2i+1}^\alpha) : i < \omega, \alpha < \kappa\}$$

is an inp-pattern of length κ, which yields the inequality $\kappa_{ict}(x = x, \emptyset) \leq \kappa_{inp}$.

□

We are now ready to present the following:

Definition 2.1.12. Let T be a (complete) theory. We say that T is *dp-finite* if the dp-rank of any 1-type is finite.

Note that if T is dp-finite, then any n-type has finite dp-rank: this is just induction on n and subadditivity of dp-rank as in Corollary 2.1.6. This in turn implies that T is dependent and $\kappa_{inp} = \kappa_{ict}(x = x, \emptyset)$ – for any single free variable x – is finite, because of Propositions 2.1.3, 2.1.8 and 2.1.11. Finally, if π is any 1-type, then $\pi \vdash x = x$ yields $\text{dp-rk}(\pi, A) \leq \text{dp-rk}(x = x, A)$. But Lemma 2.1.4 implies that $\text{dp-rk}(x = x, A) \leq \text{dp-rk}(x = x, \emptyset)$, so $\text{dp-rk}(\pi, A)$ is finite whenever $\kappa_{ict}(x = x, \emptyset)$ is finite. We have proved the following:

Proposition 2.1.13. *Let T be a (complete) theory and let x be a single variable. The following statements are equivalent:*

1. *T is dp-finite.*
2. *Every complete n-type has finite dp-rank.*
3. *T is dependent and κ_{inp} is finite.*
4. *$\kappa_{ict}(x = x, \emptyset)$ is finite.*

2.2 Valued Fields

Here we will fix some notation that will be used throughout the document.

An *ordered abelian group* is an abelian group $(\Gamma, +, 0)$ together with a linear order $\leq$ in Γ satisfying $\gamma_1 + \gamma \leq \gamma_2 + \gamma$ whenever $\gamma_1 \leq \gamma_2$, for any triple $\gamma, \gamma_1, \gamma_2 \in \Gamma$. An ordered abelian group $(\Gamma, +, 0, <)$ is called *archimedean* if for any pair γ, ε such that $\varepsilon > 0$, there is some $n \in \mathbb{N}$ satisfying $\gamma < n\varepsilon$. This is equivalent to saying that there are no non-trivial *convex* subgroups of Γ, and to saying that Γ embeds into $(\mathbb{R}, +, 0, <)$. The *rank* of an ordered abelian group Γ, denoted by $\text{rk}(\Gamma)$, is the order-type of non-trivial convex subgroups, ordered by inclusion.

Let $(K, +, 0, \cdot)$ be a field and let $(\Gamma, +, 0, <)$ be an ordered abelian group. Let ∞ be a symbol satisfying $\infty + \infty = \gamma + \infty = \infty + \gamma$ for all $\gamma \in \Gamma$.

A *valuation on* K is a surjective map $v : K \longrightarrow \Gamma \cup \{\infty\}$ such that for any $x, y \in K$,

- ✓ $v(x) = \infty$ if and only if $x = 0$,
- ✓ $v(xy) = v(x) + v(y)$,
- ✓ $v(x + y) \geq \min\{v(x), v(y)\}$.

These are the usual axioms of a non-archimedean valuation on $\mathbb{R}$, but generalized to admit any ordered abelian group as target set. We call (K, v) a *valued field*, and we call Γ its *value group*. Define $\mathcal{O}$ to be the ring $\{x \in K^\times : v(x) \geq 0\}$. We call $\mathcal{O}$ the *valuation ring of* (K, v). The complement in $\mathcal{O}$ of $\mathcal{O}^\times$ is an ideal, and is referred to as $\mathscr{M}$. Note that $\mathcal{O}^\times = \{x \in \mathcal{O} : v(x) = 0\}$, so that $\mathscr{M} = \{x \in \mathcal{O} : v(x) > 0\}$. The ideal $\mathscr{M}$ is maximal, and the field ${}^{\mathcal{O}}\!/_{\mathscr{M}}$ is called the *residue field* of (K, v), denoted by k. The residue map from $\mathcal{O}$ to k is often denoted by $\text{res}(x)$, $\overline{x}$ or $x + \mathscr{M}$. If $f = \sum_{i=0}^{n} a_i X^i$ is a polynomial in $\mathcal{O}[X]$, we define $\overline{f}$ to be the polynomial $\sum_{i=0}^{n} \overline{a_i} X^i$ in $k[X]$.

We will frequently add a subscript in order to emphasize the dependence to v, i.e. $\mathcal{O} = \mathcal{O}_v$, $\Gamma = \Gamma_v$, $k = k_v$ and $\mathscr{M} = \mathscr{M}_v$.

It can be seen that the definition of $\mathcal{O}$ implies that for any $x \in K^\times$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Any integral domain satisfying this condition (when considered as a subring of its field of fractions) is also called a *valuation ring*.

If R is a valuation ring, then there is a valuation on $K = \mathrm{Quot}(R)$ having R as its valuation ring. Indeed,

✓ $\Gamma := {K^\times}/{R^\times}$ *is an ordered abelian group.* Define $xR^\times \geq R^\times$ if and only if $x \in R$. If $x \in R$ and $xy^{-1} \in R^\times$, then $y = (yx^{-1})(x) \in R$, so the order is well defined. Also, it is linear because for any $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$ implies that either $xR^\times \geq R^\times$ or $x^{-1}R^\times \geq R^\times$, which is equivalent to $R^\times \geq xR^\times$.

✓ $v : K^\times \longrightarrow \Gamma$ *defined by* $v(x) = xR^\times$ *is a valuation on* K. We have to show that $v(x + y) \geq \min\{v(x), v(y)\}$. Suppose that $xR^\times \leq yR^\times$, so that $\min\{v(x), v(y)\} = xR^\times$. We must then argue that $\frac{x+y}{x} \in R$. But $\frac{x+y}{x} = 1 + \frac{y}{x}$ and, by hypothesis, $\frac{y}{x} \in R$.

✓ $\mathcal{O}_v = R$. This follows by definition of v : if $x \in K^\times$ satisfies that $v(x) \geq 0$ then, by definition, $xR^\times \geq R^\times$, which is equivalent to $x \in R$.

Thus, whenever we refer to $(K, \mathcal{O})$ as a valued field, we mean that $\mathcal{O}$ is a valuation ring having K as its field of fractions, and that the valuation v on K we mean to use is the one constructed above, so that $\mathcal{O}_v = \mathcal{O}$. Also, the *rank* of $\mathcal{O} = \mathcal{O}_v$ is defined to be the rank of Γ_v.

Let L/K be an extension of fields, w be a valuation on L and v a valuation on K. We say that (L, w) *extends* (K, v), denoted by $(L, w) \supseteq (K, v)$, if and only of $w \restriction_K = v$. This is equivalent to saying that $\mathcal{O}_w \cap K = \mathcal{O}_v$. Theorem 3.1.1 of [5], known as Chevalley's theorem, implies that for any valued field (K, v) and any field extension L/K, there is at least one valuation w on L extending v. It can be shown that if $(L, w) \supseteq (K, v)$ then k_v is a subfield of k_w and Γ_v is a subgroup of Γ_w that satisfy $[L : K] \geq [k_w : k_v][\Gamma_w : \Gamma_v]$. An extension $(L, w) \supseteq (K, v)$ is called *immediate* if $[k_w : k_v] = [\Gamma_w : \Gamma_v] = 1$.

Proposition 2.2.1. *Let $(L, w) \supseteq (K, v)$ be an extension of valued fields, with L/K algebraic. Then $k_w \subseteq k_v^{alg}$ and $\Gamma_w \subseteq \mathrm{Div}(\Gamma_v)$.*[1] *Moreover, if K is algebraically closed, then k_v is algebraically closed and Γ_v is divisible.*

Proof. Let $x + \mathscr{M} \in k_w$ for some $x \in \mathcal{O}_w$. Then there is some non-zero $p \in K[X]$ such that $p(x) = 0$. By multiplying by the inverses of the coefficients of p which are not elements of $\mathcal{O}_v$, we get that $q(x) = 0$ for some non-zero $q \in \mathcal{O}_v[X]$. Therefore $\overline{q}(\overline{x}) = \overline{0}$ as wanted.

Now suppose $w(x) \in \Gamma_w$ for some non-zero $x \in L$. Suppose $\sum_{i=0}^n a_i x^i = 0$ for some coefficients $a_0, \ldots, a_n \in K$. Out of the non-zero coefficients, there must be a pair of indices $i < j$ such that $w(a_i x^i) = w(a_j x^j)$, because otherwise

$$\infty = w(\sum_{i=0}^n a_i x^i) = w(a_k x^k) = w(a_k) + k w(x)$$

for some k,[2] which is absurd. Therefore $(j - i)w(x) = w(a_i/a_j)$, as wanted.

Finally, if K is algebraically closed, the same arguments (taking $x \in K$ rather than $x \in L$) yield that k_v is algebraically closed and that Γ_v is divisible. $\square$

[1] $\mathrm{Div}(\Gamma)$ stands for the *divisible hull* of Γ, i.e. the smallest divisible abelian group containing Γ.

[2] An easy computation yields that $v(a + b) = v(a)$ whenever $v(b) > v(a)$ for any a, b in any valued field (K, v).

If L/K is algebraic and the separable degree $n = [L : K]_s$ is finite, then it bounds the number of extensions of v to L. If n is finite, the *fundamental inequality* (cf. Theorem 3.3.4 of [5]) states that

$$[L : K] \geq \sum_{i=1}^{n} [k_{w_i} : k_v][\Gamma_{w_i} : \Gamma_v],$$

where $\{w_1, \dots, w_n\}$ is the set of all extensions of v to L.

In what follows, we will need two important theorems of valuation theory. They are Theorem 3.2.7 and Theorem 3.2.15 of [5] respectively.

Fact 2.2.2 (Weak Approximation Theorem, cf.Theorem 3.2.7 of [5])**.** *Let $\mathcal{O}_1, \dots, \mathcal{O}_n$ be an $\subseteq$-antichain of valuation rings of a field K, with maximal ideals $\mathscr{M}_1, \dots, \mathscr{M}_n$ respectively, and let*

$$\mathfrak{p}_i = \bigcap_{j=1}^{n} \mathcal{O}_j \cap \mathscr{M}_i.$$

Then $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is an $\subseteq$-antichain and the set of all maximal ideals of $\bigcap_{j=1}^{n} \mathcal{O}_j$. Furthermore, for any tuple $(a_1, \dots, a_n) \in \mathcal{O}_1 \times \dots \times \mathcal{O}_n$, there is some $\beta \in \bigcap_{j=1}^{n} \mathcal{O}_j$ such that $\beta - a_i \in \mathscr{M}_i$ for every $i \in \{1, \dots, n\}$.

Fact 2.2.3 (Conjugation Theorem, cf. Theorem 3.2.15 of [5])**.** *Suppose L/K is a normal extension of fields. Let $\mathcal{O}$ be a valuation ring of K and let $\mathcal{O}', \mathcal{O}''$ be two valuation rings of L extending $\mathcal{O}$. Then there is some $\sigma \in \mathrm{Aut}(L/K)$ such that $\sigma(\mathcal{O}') = \mathcal{O}''$.*

2.3 Henselianity

This section will follow very closely the presentation of Hensel's Lemma in Section 4 of [5].

The proof of the following lemma is a straight computation.

Lemma 2.3.1. *Let $(K, \mathcal{O})$ be a valued field and let $f = \sum_{i=1}^{n} a_i X^i \in K[X]$. Then*

$$f(X + Y) = \sum_{i=0}^{n} f_i(X) Y^i$$

with

$$f_i(X) = \sum_{j=i}^{n} a_j \binom{j}{i} X^{j-i}.$$

In particular $f_0 = f$ and $f_1 = f'$. If $f \in \mathcal{O}[X]$ then $f_i \in \mathcal{O}[X]$ for every $i \in \{1, \dots, n\}$.

Lemma 2.3.2. *Suppose $(K, \mathcal{O}) \subseteq (N, \mathcal{O}^*)$ is an extension of valued fields, where N/K is a finite Galois extension. Then there exists a subfield L of N containing K such that $(K, \mathcal{O}) \subseteq (L, \mathcal{O}^* \cap L)$ is immediate and $(L, \mathcal{O}^* \cap L) \subseteq (N, \mathcal{O}^*)$ is the only possible extension.*

Proof. Let $\mathcal{O}^* = \mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$ be the collection of all the different extensions of $\mathcal{O}$ to N. Let $H = \{\sigma \in \mathrm{Aut}(N/K) : \sigma(\mathcal{O}^*) = \mathcal{O}^*\}$ and let L be the fixed field of H. The rings $\mathcal{O}_1 \cap L, \dots, \mathcal{O}_m \cap L$ are all the extensions of $\mathcal{O}$ to L, but they may not necessarily be pairwise distinct. If $\mathcal{O}_j$ extends $\mathcal{O}^* \cap L$, then, by the Conjugation Theorem 2.2.3, there exists an automorphism $\sigma \in \mathrm{Aut}(N/L)$ such that $\sigma(\mathcal{O}^*) = \mathcal{O}_j$. Since N/K is Galois, $\mathrm{Aut}(N/L)$ is exactly $H = \{\sigma \in \mathrm{Aut}(N/K) : \sigma(\mathcal{O}^*) = \mathcal{O}^*\}$, therefore $\mathcal{O}_j = \mathcal{O}^*$.

Now, let us see that $(L, \mathcal{O}^* \cap L)$ is an immediate extension of $(K, \mathcal{O})$.

✓ In order to prove that the residue class field of $(L, \mathcal{O}^* \cap L)$ is the same as $(K, \mathcal{O})$, it is sufficient to prove that for any $\alpha \in \mathcal{O}^* \cap L$ there exists some $a \in \mathcal{O}$ such that $\alpha - a$ is an element of $\mathscr{M}_1$, the maximal ideal of $\mathcal{O}^* \cap L$. Use the Weak Approximation Theorem 2.2.2 to find some $\beta \in \bigcap_{i=1}^m \mathcal{O}_j \cap L$ such that $\alpha - \beta \in \mathscr{M}_1$ and $\beta \in \mathscr{M}_j$, the maximal ideal of $\mathcal{O}_j \cap L$, for any $j \in \{2, \ldots, m\}$.

Let $\beta = \beta_1, \ldots, \beta_n$ be the set of all distinct K-conjugates of β in N. Then $a = -\beta_1 - \ldots - \beta_n$ is an element of K, and $\alpha + a = (\alpha - \beta) - \beta_2 - \ldots - \beta_n \in \mathscr{M}_1$. Indeed, by the choice of β, $\alpha - \beta \in \mathscr{M}_1$. Also, if $\beta_j \neq \beta_1$, then there is some $\tau \in \mathrm{Aut}(N/K)$ such that $\tau(\beta) = \beta_j$. Then $\tau \notin H$, because otherwise it would fix β. Hence $\mathcal{O}^* = \tau(\mathcal{O}_j)$ for some $j \in \{2, \ldots, m\}$, so $\beta_j = \tau(\beta) \in \mathscr{M}_1$. Since $\beta_j \in \mathcal{O}^*$ for any $j \in \{1, \ldots, m\}$, then $a \in \mathcal{O}$ as wanted.

✓ In order to prove that the value group of $(L, \mathcal{O}^* \cap L)$ is the same as $(K, \mathcal{O})$, it is sufficient to prove that for any $\alpha \in L^\times$ there exists some $a \in K^\times$ such that $w(\alpha) = w(a)$, if w is the valuation associated to $\mathcal{O}^*$ in N. To this end, use the Weak Approximation Theorem 2.2.2 to find some $\beta \in \bigcap_{i=1}^m \mathcal{O}_j \cap L$ such that $\beta - 1 \in \mathscr{M}_1$ and $\beta \in \mathscr{M}_j$ for any $j \in \{2, \ldots, m\}$.

Since $\beta - 1 \in \mathscr{M}_1$, then $w(\beta - 1) > 0 = w(1)$, so $w(\beta) = w(\beta - 1 + 1) = w(1) = 0$, i.e., $\beta \in (\mathcal{O}^*)^\times$. Hence, if $\tau \in H$, then $\tau(\beta) \in (\mathcal{O}^*)^\times$, i.e., $w(\tau(\beta)) = 0$. If, however, $\tau \in \mathrm{Aut}(N/K) \setminus H$, then $\tau^{-1} \notin H$ and $\tau(\mathcal{O}_j) = \mathcal{O}^*$ for some $j \in \{2, \ldots, m\}$.

Since $\beta \in \mathscr{M}_j$, then $\tau(\beta) \in \mathscr{M}_1$, this is, $w(\tau(\beta)) > 0$. Now, choose some $\nu \in \mathbb{Z}$ such that $w(\beta^\nu \alpha) \neq w(\tau(\beta^\nu \alpha))$ for any $\tau \in \mathrm{Aut}(N/K) \setminus H$[3]. Let $\beta^\nu \alpha = \alpha_1, \ldots, \alpha_n$ be the set of all distinct K-conjugates of $\beta^\nu \alpha$ in N. We then know that for any $t \in \{1, \ldots, n\}$ the element

$$a_t = (-1)^t \sum_{1 \leq i_1 < \ldots < i_t \leq n} \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_t}$$

lies in K, and that if $\alpha_j = \tau(\alpha_1)$ for some $j \in \{2, \ldots, n\}$ – i.e., if $\tau \notin H$ – then $w(\alpha_j) = w(\tau(\beta^\nu \alpha)) \neq w(\beta^\nu \alpha) = w(\alpha_1)$.

Suppose that $j_1 < \ldots < j_r$ are the indices that satisfy $w(\alpha_j) < w(\alpha_1)$. If there are no such indices, then $w(\alpha_1) < w(\alpha_j)$ for any $j \in \{2, \ldots, n\}$ and $w(a_1) = w(\alpha_1 + \ldots + \alpha_n) = w(\alpha_1) = w(\alpha)$. Note that $w(\alpha_{j_1} \cdot \ldots \cdot \alpha_{j_r}) = w(\alpha_{j_1}) + \ldots + w(\alpha_{j_r}) < w(\alpha_{j_1} \cdot \ldots \cdot \alpha_{j_{r-1}} \cdot \alpha_i)$ for any $i \notin \{j_1, \ldots, j_r\}$, so that

$$w(a_r) = w(\alpha_{j_1} \cdot \ldots \cdot \alpha_{j_r}).$$

Similarly, if $i \notin \{1, j_1, \ldots, j_r\}$, then $w(\alpha_1 \cdot \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_r}) = w(\alpha_1) + w(\alpha_{j_1}) + \ldots + w(\alpha_{j_r}) < w(\alpha_1 \cdot \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_{r-1}} \cdot \alpha_i)$, so

$$w(a_{r+1}) = w(\alpha_1 \cdot \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_r}).$$

This implies that $w(a_{r+1}/a_r) = w(\alpha_1) = w(\alpha)$, as wanted.

□

[3]This can be achieved because if $\tau_1, \ldots, \tau_n$ are all the elements of $\mathrm{Aut}(N/K) \setminus H$, then the set

$$\{\nu w(\tau_i(\beta)) : \nu \in \mathbb{Z}, i \in \{1, \ldots, n\}\} \setminus \{w\left(\frac{\alpha}{\tau_i(\alpha)}\right) : i \in \{1, \ldots, n\}\}$$

is infinite.

Definition 2.3.3. A valued field $(K, \mathcal{O})$ is called *henselian* if $\mathcal{O}$ has a unique prolongation[4] to every algebraic extension L of K.

Note that $(K, \mathcal{O})$ is henselian if and only if $\mathcal{O}$ extends uniquely to any finite extension L of K. Also, if $(K, \mathcal{O})$ is henselian, then $\mathcal{O}$ extends uniquely to the separable closure K^s of K by definition. This is in fact sufficient, because if L is a finite extension of K, the extension can be decomposed into $L/K^s \cap L$ and $K^s \cap L/K$. By hypothesis, the latter admits a unique extension $\mathcal{O}^*$ of $\mathcal{O}$, as does the former with $\mathcal{O}^*$, for it is a purely inseparable extension.

Observation 2.3.4. Let (K, v) be a valued field. A polynomial $f = \sum_{i=0}^{n} a_i X^i \in K[X]$ with coefficients in K is *primitive* if its *Gauss valuation* $w(f) = \min_{i=0}^{n} v(a_i)$ equals zero. Every primitive polynomial is an element of $\mathcal{O}[X]$. If $f \in K[X]$ and $w(f) = v(a)$ for some coefficient a of f, then $w(a^{-1}f) = w(f) - w(a) = w(f) - v(a) = 0$, so $f = a(a^{-1}f) = af^*$ for some primitive polynomial $f^* = a^{-1}f$. This implies that if $f \in \mathcal{O}[X]$ decomposes as $g_1 \dots g_n$ with irreducible factors $g_1, \dots, g_n \in K[X]$, then there are some primitive polynomials $h_1, \dots, h_n$ irreducible in $K[X]$ such that $f = h_1 \dots h_n$. Indeed, let $f = af^*$ and $g_i = b_i g_i^*, i \in \{1, \dots, n\}$, for some primitive polynomials $f^*, g_1^*, \dots, g_n^*$. Then $af^* = b_1 \dots b_n g_1^* \dots g_n^*$, and

$$0 \leq v(a) = w(a) + w(f^*) = w(b_1 \dots b_n g_1^* \dots g_n^*) = v(b_1 \dots b_n),$$

so $b_1 \dots b_n g_1^*, g_2^*, \dots, g_n^* \in \mathcal{O}[X]$ remain to be irreducible in $K[X]$.

The driving idea of the following theorem is that whenever an irreducible polynomial $f \in K[X]$ splits completely in a Galois extension N of $(K, \mathcal{O})$, then its roots are all K-conjugate. If $(K, \mathcal{O})$ is henselian, then the valuation associated to the unique extension of $\mathcal{O}$ to N is constant on the roots of f, because conjugating the valuation yields another valuation, which is unique by henselianity.

Theorem 2.3.5 (Cf. Theorem 4.1.3 of [5])**.** *Let $(K, \mathcal{O})$ be a valued field. Let v be the associated valuation of $\mathcal{O}$ and let $\mathscr{M}$ be its maximal ideal. The following statements are equivalent:*

1. *$(K, \mathcal{O})$ is henselian,*
2. *For every polynomial $f \in \mathcal{O}[X]$ irreducible in $K[X]$, either $\deg(\overline{f}) = 0, \deg(f) = 1$ or all the roots of $\overline{f}$ are multiple,*
3. ***Lifting Simple Roots:** For every $f \in \mathcal{O}[X]$ and every $a \in \mathcal{O}$, if $\overline{a}$ is a simple root of $\overline{f}$, then there is some $\alpha \in \mathcal{O}$ such that $\overline{\alpha} = \overline{a}$ and $f(\alpha) = 0$,*
4. ***Hensel's Lemma:** For every $f \in \mathcal{O}[X]$ and every $a \in \mathcal{O}$ such that $v(f(a)) > 2v(f'(a))$, there is some $\alpha \in \mathcal{O}$ such that $v(\alpha - a) > v(f'(a))$ and $f(\alpha) = 0$,*
5. *Every separable polynomial in $\mathcal{O}[X]$, irreducible in $K[X]$, of the form $X^n + a_1 X^{n-1} + \dots + a_n$, where $a_1 \notin \mathscr{M}$ and $a_i \in \mathscr{M}$ for any $i \in \{2, \dots, n\}$, has a root in $\mathcal{O}$.*

Proof. 1. Let us start with a henselian valued field $(K, \mathcal{O}), \mathcal{O} = \mathcal{O}_v$, and an irreducible polynomial $f \in \mathcal{O}[X]$ of degree n. Suppose that $\overline{f} \notin k_v$. If $\widetilde{K}$ is an algebraic closure of K, then $\mathcal{O}$ extends uniquely to, say, $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}_{\widetilde{v}}$. In this case, f can be written as $\prod_{i=1}^{n}(aX - x_i)$ for some $a, x_1, \dots, x_n \in \widetilde{K}$. Since $a^n \in \mathcal{O}$ is the principal coefficient of

[4] The words *prolongation* and *extension* are used as synonyms.

f, then $\widetilde{v}(a) = \frac{\widetilde{v}(a^n)}{n} \geq 0$ and also $f(0) = (-1)^n x_1 \dots x_n \in \mathcal{O}$. If $x_i \neq x_j$, then there is some $\sigma \in \mathrm{Aut}(\widetilde{K}/K)$ such that $\sigma\left(\frac{x_i}{a}\right) = \frac{x_j}{a}$. Since $\sigma(\widetilde{\mathcal{O}})$ extends $\mathcal{O}$, then $\sigma(\widetilde{\mathcal{O}}) = \widetilde{\mathcal{O}}$ and, consequently, $\widetilde{v} \circ \sigma = \widetilde{v}$ and $\sigma(\widetilde{\mathscr{M}}) = \widetilde{\mathscr{M}}$, where $\widetilde{\mathscr{M}}$ is the maximal ideal of $\widetilde{\mathcal{O}}$. This implies that $\widetilde{v}\left(\frac{x_i}{a}\right) = \widetilde{v}\left(\frac{x_i}{a}\right)$, i.e., $\widetilde{v}$ is constant on the $\frac{x_i}{a}$'s and, accordingly, on the x_i's. Call the latter constant value δ. Then

$$0 \leq v(x_1 \dots x_n) = \widetilde{v}(x_1 \dots x_n) = n\delta.$$

If $\delta > 0$ then $x_i \in \widetilde{\mathscr{M}}$ for any i, so $\overline{f} = (\overline{a}X)^n$, i.e., $\overline{f}$ is a power of an irreducible polynomial $\overline{g} = \overline{a}^n X$ such that $g = a^n X$ is primitive. In this case either $n = 1$ or $\overline{f}$ cannot have simple roots.

The latter is still true if $\delta = 0$, in which $\overline{\frac{x_i}{a}} \neq \overline{0}$ is a root of $\overline{f}$ for any i. Indeed, if $n \geq 2$, $\overline{g}$ divides $\overline{f}$ for some polynomial $g \in \mathcal{O}[X]$, and if $\overline{\frac{x_i}{a}}$ is a root of $\overline{g}$, then for any other root (whose existence is granted by n being at least 2) $\overline{\frac{x_j}{a}}$ of $\overline{f}$ there would exist some $\sigma \in \mathrm{Aut}(\widetilde{K}/K)$ such that $\sigma\left(\frac{x_i}{a}\right) = \frac{x_j}{a}$. Since $\overline{g}\left(\overline{\frac{x_i}{a}}\right) = 0$, then $g(\frac{x_i}{a}) \in \widetilde{\mathscr{M}}$ and

$$g\left(\frac{x_j}{a}\right) = g\left(\sigma\left(\frac{x_i}{a}\right)\right) = \sigma\left(g\left(\frac{x_i}{a}\right)\right) \in \sigma(\widetilde{\mathscr{M}}) = \widetilde{\mathscr{M}},$$

implying that $\overline{\frac{x_j}{a}}$ is also a root of $\overline{g}$. In case $\overline{g}$ is $X - \overline{\frac{x_1}{a}}$, then its root is multiple in $\overline{f}$, so $\overline{f}$ itself cannot have simple roots whenever $\deg(f) \geq 2$.

2. Item number 1 allows us to "lift simple roots" in the following sense: let $f \in \mathcal{O}[X]$ and $a \in \mathcal{O}$ be such that $\overline{a}$ is a simple root of $\overline{f}$. If $f_1 \dots f_m$ is a factorization into irreducibles of f in $\mathcal{O}[X]$, then, say, $\overline{f_1}(\overline{a}) = 0$. Since f_1 is irreducible, either $\overline{a}$ is a multiple root of $\overline{f_1}$ (hence of $\overline{f}$) or $\deg(f_1) = 1$, because of the argument above. Since $\overline{a}$ is a simple root, then f_1 is a linear factor of f, i.e., f has a root. In particular, if $f_1 = e(X - b)$, then $\overline{e}(X - \overline{b}) = X - \overline{a}$, so $\overline{e} = 1$, $\overline{b} = \overline{a}$ and b is a root of f.

3. We are ready to prove Hensel's Lemma for henselian valued fields. Let $f \in \mathcal{O}[X]$ and let $a \in \mathcal{O}$ be such that $v(f(a)) > 2v(f'(a))$, where v is the associated valuation of $\mathcal{O}$. The quadratic approximation of f at a given by Lemma 2.3.1 yields the expression

$$f(a - X) = f(a) - Xf'(a) + X^2 g(X)$$

for some polynomial $g \in \mathcal{O}[X]$. Since $v\left(\frac{f(a)}{f'(a)^2}\right) > 0$ (recall that $f'(a)$ is invertible because its valuation is not infinite), we would like to divide by $f'(a)^2$ in order to reduce modulo $\mathscr{M}$. Indeed, write

$$\frac{f(a - X)}{f'(a)^2} = \frac{f(a)}{f'(a)^2} - \frac{X}{f'(a)} + \frac{X^2}{f'(a)^2} g(X),$$

and let $Y = \frac{X}{f'(a)}$ so that the above equation equals

$$\frac{f(a - f'(a)Y)}{f'(a)^2} = \frac{f(a)}{f'(a)^2} - Y + Y^2 g(f'(a)Y).$$

Reducing modulo $\mathscr{M}$ yields

$$\frac{\overline{f}(\overline{a} - \overline{f'(a)}Y)}{\overline{f'(a)^2}} = Y(Y\overline{g}(\overline{f'(a)}Y) - 1),$$

which has $\overline{0}$ as a simple root. Lifting roots to $\mathcal{O}$ – using item number 2 – implies that $\frac{f(a-f'(a)Y)}{f'(a)^2}$ has a root $b \in \mathcal{O}$ such that $\overline{b} = \overline{0}$, i.e. $b \in \mathscr{M}$, so $a - f'(a)b$ is a root of f. Also

$$v(a - (a - f'(a)b)) = v(f'(a)b) = v(f'(a)) + v(b) > v(f'(a)).$$

4. Now, take for example a polynomial $f = X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n \in \mathcal{O}[X]$. Then, in order to get a simple root after reducing modulo $\mathscr{M}$, one can require that $\overline{a_i} = 0$ for any $i \in \{2, \ldots, n\}$ and that $\overline{a_1} \neq 0$. This is, if $a_1 \notin \mathscr{M}$ and $a_2, \ldots, a_n \in \mathscr{M}$, then
$$\overline{f} = X^n + \overline{a_1}X^{n-1} = X^{n-1}(X + \overline{a_1}),$$
which has $-\overline{a_1}$ as a simple root. Note that $f(-a_1) \in \mathscr{M}$ and $f'(-a_1) = \pm a_1^{n-1} + b$ for some $b \in \mathscr{M}$. Hence $v(f(-a_1)) > 0 = 2v(f'(-a_1)^2)$, so by Hensel's lemma – i.e., item number 3 –, f must have a root.

5. This last item in turn implies henselianity of $(K, \mathcal{O})$. Suppose otherwise, yielding a finite extension N of K in which $\mathcal{O}$ has at least two extensions. Without loss, N/K is Galois (by taking the Galois closure of N/K and extending the different extensions of $\mathcal{O}$ to this closure). With the notation of Lemma 2.3.2, find some $\beta \in \bigcap_{i=1}^m \mathcal{O}_i \cap L$ such that $\beta - 1 \in \mathscr{M}_1$ and $\beta \in \mathscr{M}_j$ for any $j \in \{2, \ldots, m\}$. Since $m \geq 2$, $\beta \notin \mathscr{M}_1$ and $\beta \in \mathscr{M}_2$, then β cannot be fixed by the K-automorphism taking $\mathcal{O}_1$ to $\mathcal{O}_2$, so $\beta \notin \mathrm{Fix}(\mathrm{Aut}(N/K)) = K$. Therefore, since the extension is Galois, its minimal polynomial $f = X^n + a_1X^{n-1} + \ldots + a_n \in K[X]$ is separable and cannot have a root in K. But according to (the proof of) Lemma 2.3.2, since $\beta - 1 \in \mathscr{M}_1$, then $1 + a_1 \in \mathscr{M}_1$ and $a_j \in \mathscr{M}_1$ for any other $j \in \{2, \ldots, n\}$, contradicting item number 4. $\square$

We already know that if (K, v) is henselian and $(L, w) \supseteq (K, v)$ is an algebraic extension, then (L, w) is also henselian. If, instead, $K \subseteq L$ and (L, v) is henselian, then by item 5 of Theorem 2.3.5, $(K, v\restriction_K)$ is henselian if and only if for any separable polynomial in $(\mathcal{O}_v \cap K)[X]$ of the form $X^n + a_1X^{n-1} + \ldots + a_n$, where $a_1 \notin \mathscr{M}_v \cap K$ and $a_i \in \mathscr{M}_v \cap K$ for any $i \in \{2, \ldots, n\}$, has a root in $\mathcal{O}_v \cap K$. Such a polynomial has already a root in $\mathcal{O}_v$ because of henselianity of (L, v), so if K is separably closed in L, then such a root must lie in K. This is,

Corollary 2.3.6. *Let (L, v) be a henselian valued field and let K be a subfield of L. If K is relatively separably closed in L, then $(K, v\restriction_K)$ is henselian.*

We will use the following

Fact 2.3.7 (Cf. Theorem 5.2.2. of [5])**.** *Any valued field $(K, \mathcal{O})$ has a henselian immediate extension $(K^h, \mathcal{O}^h)$ which is minimal in the sense that for any other henselian extension $(L, \mathcal{O}')$ there is some (uniquely determined) K-embedding $\iota : (K^h, \mathcal{O}^h) \longrightarrow (L, \mathcal{O}')$.*

2.3.1 Hahn Series Fields and Kaplansky Theory

In this section we will introduce and prove some basic facts about Hahn series fields and some related results about Kaplansky Theory. We will propose a solution to exercise 3.5.6 of [5], and follow closely some results of [9] which will introduce us to the model theory of valued fields.

Let (K, v) be a non-trivial valued field and let $\alpha, \beta, \gamma, \nu, \lambda, \rho$ be ordinal numbers. A sequence $s = (a_\nu)_{\nu<\rho}$ – with ρ being a limit ordinal – of elements of K is called a *pseudo-Cauchy sequence* if for all $\alpha < \beta < \gamma < \rho$ we have

$$v(a_\gamma - a_\beta) > v(a_\beta - a_\alpha).$$

An element $b \in K$ is called a *pseudo-limit of s* if for all $\nu < \rho$ we have

$$v(b - a_\nu) = v(a_{\nu+1} - a_\nu).$$

Note that b is a pseudo-limit of s if and only of the sequence $(v(b - a_\nu))_{\nu<\rho}$ is strictly increasing. We call (K, v) *pseudo-complete*, also known as *spherically complete*, if every pseudo-Cauchy sequence of K has a pseudo-limit in K.

Pseudo-complete valued fields are examples of *maximal* valued fields, i.e., fields with no proper immediate extensions. Indeed, assume that (K', v') is an immediate extension of (K, v) with $K' = K(z)$ for some $z \in K' \setminus K$. Let $\Delta = \{v'(z - a) : a \in K\} \subseteq \Gamma' = \Gamma$. Note that Δ cannot have a greatest element. This is because if $v'(z - a)$ is any element of Δ, then $v'(z - a) = v(b)$ for some $b \in K$ – because $\Gamma' = \Gamma$ –, so $v'\left(\frac{z-a}{b}\right) = 0$. Choose some $d \in K$ such that $v'\left(\frac{z-a}{b} - d\right) > 0$ – using the fact that $\overline{K'} = \overline{K}$ –, so that $a + bd \in K$ and $v'(z - a - bd) > v(a) = v'(z - a)$. Therefore Δ has some (infinite) cofinality ρ. Let $(\delta_\nu)_{\nu<\rho}$ be a cofinal sequence of Δ and let $(a_\nu)_{\nu<\rho} \subseteq K$ be such that $v'(z - a_\nu) = \delta_\nu$. If $\alpha < \beta < \rho$, then

$$v'(z - a_\beta) = \delta_\beta > \delta_\alpha = v'(z - a_\alpha),$$

so

$$v(a_\beta - a_\alpha) = v'(a_\beta - z + z - a_\alpha) = v'(z - a_\alpha) = \delta_\alpha.$$

This implies that $(a_\nu)_{\nu<\rho}$ is pseudo-Cauchy, because if $\alpha < \beta < \gamma < \rho$, then

$$v(a_\gamma - a_\beta) = \delta_\beta > \delta_\alpha = v(a_\beta - a_\alpha).$$

However, $(a_\nu)_{\nu<\rho}$ cannot have any pseudo-limit in K, because if $a \in K$ were some pseudo-limit of the sequence, then there would be some $\nu < \rho$ such that $v'(z-a) < \delta_\nu = v(z-a_\nu)$, hence

$$\delta_\nu = v(a_{\nu+1} - a_\nu) = v(a - a_\nu) = v'(a - z + z - a_\nu) = v'(z - a),$$

which is absurd. We have proved the following propositions.

Proposition 2.3.8. *Let (K, v) be a valued field.*

1. *If $(v(z - a_\nu))_{\nu<\rho}$ is strictly increasing, then $(a_\nu)_{\nu<\rho}$ is a pseudo-Cauchy sequence having z as a pseudo-limit.*

2. *If $(K', v') \supseteq (K, v)$ is a proper immediate extension, then for any $z \in K' \setminus K$ there is a pseudo-Cauchy sequence $(a_\nu)_{\nu<\rho}$ in K having z as pseudo-limit but without none in K. In particular, any pseudo-complete valued field is maximal.*

Maximal valued fields are examples of henselian valued fields, because of Lemma 2.3.2: if $(K, \mathcal{O})$ is maximal and $(K', \mathcal{O}')$ is any finite extension of $(K, \mathcal{O})$, then its Galois closure $(N, \widetilde{\mathcal{O}})$ is also a finite extension of $(K, \mathcal{O})$, unique under maximality of $(K, \mathcal{O})$. Hence $(K', \mathcal{O}')$ is also a unique extension of $(K, \mathcal{O})$.

The next definition will be used throughout the rest of this document.

Definition 2.3.9. Let K be a field and Γ be an ordered abelian group. We define the *Hahn series field* $K(\!(t^\Gamma)\!)$, also called the *Formal Power series field*, as the set of all formal sums

$$f = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$$

where $a_\gamma \in K$ for any $\gamma \in \Gamma$, t is a symbol, and the set

$$\mathrm{Supp}(f) := \{\gamma \in \Gamma : a_\gamma \neq 0\}$$

is well-ordered.

We can endow $K(\!(t^\Gamma)\!)$ with a natural addition by defining

$$\left(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma\right) + \left(\sum_{\gamma\in\Gamma} b_\gamma t^\gamma\right) := \sum_{\gamma\in\Gamma}(a_\gamma + b_\gamma)t^\gamma,$$

and a natural multiplication defined by

$$\left(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma\right) \cdot \left(\sum_{\gamma\in\Gamma} b_\gamma t^\gamma\right) := \sum_{\gamma\in\Gamma}\left(\sum_{\delta+\varepsilon=\gamma} a_\delta b_\varepsilon\right) t^\gamma.$$

These operations are well defined: let $f = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ and $g = \sum_{\gamma\in\Gamma} b_\gamma t^\gamma$. If S is a non-empty subset of $\mathrm{Supp}(f+g)$ and $\gamma \in S$, then $a_\gamma + b_\gamma \neq 0$, so either $a_\gamma \neq 0$ or $b_\gamma \neq 0$. Then either $\gamma \in S \cap \mathrm{Supp}(f)$ or $\gamma \in S \cap \mathrm{Supp}(g)$, so $\min S$ equals

$$\min\{\min(S \cap \mathrm{Supp}(f)), \min(S \cap \mathrm{Supp}(g))\}.$$

Also, if $S \subseteq \mathrm{Supp}(fg)$ and $\gamma \in S$, then

$$\sum_{\delta+\varepsilon=\gamma} a_\delta b_\varepsilon = \sum_{\varepsilon\in\Gamma} a_{\gamma-\varepsilon} b_\varepsilon$$

has finitely many non-zero summands, otherwise there would be infinitely many $\varepsilon \in \Gamma$ such that $a_{\gamma-\varepsilon} \neq 0$ and $b_\varepsilon \neq 0$, and out of such ε's there is either an infinite increasing subsequence, yielding an infinite descending chain in $\mathrm{Supp}(f)$, or an infinite decreasing subsequence, yielding an infinite descending chain in $\mathrm{Supp}(g)$. If $(\gamma_i)_{i<\omega}$ is an infinite descending chain of $\mathrm{Supp}(fg)$, then for each γ_i there are some δ_i, ε_i such that $\delta_i + \varepsilon_i = \gamma_i$, $\delta_i \in \mathrm{Supp}(f)$ and $\varepsilon_i \in \mathrm{Supp}(g)$. If $i < j$ then $\delta_i + \varepsilon_i = \gamma_i > \gamma_j = \delta_j + \varepsilon_j$, so either $\delta_i > \delta_j$ or $\varepsilon_i > \varepsilon_j$. Define $f : [\omega]^2 \longrightarrow \{0,1\}$ to be

$$f(\{i,j\}) = \begin{cases} 0 & \text{if } i<j \text{ implies } \delta_i > \delta_j, \\ 1 & \text{otherwise.} \end{cases}$$

By Ramsey's theorem, there is an infinite set A of ω such that f is constant on $[A]^2$. If f equals 0 in $[A]^2$, then the sequence $(\delta_i : i < \omega, i \in A)$ is strictly decreasing, which is absurd. If f equals 1 in $[A]^2$, then the sequence $(\varepsilon_i : i < \omega, i \in A)$ is strictly decreasing, which is also absurd.

These operations turn $K(\!(t^\Gamma)\!)$ into a commutative ring with unity. In fact, it is a field. This will be clear after defining the following function, which will serve as a natural valuation on $K(\!(t^\Gamma)\!)$.

Define the function $v : K(\!(t^\Gamma)\!) \longrightarrow \Gamma \cup \{\infty\}$ by $v(0) = \infty$ and $v(f) = \min(\mathrm{Supp}(f))$. Then v is a valuation on $K(\!(t^\Gamma)\!)$. Indeed, if $\gamma_0 \in \Gamma$, the series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ given by $a_\gamma = 1$ if $\gamma = \gamma_0$ and $a_\gamma = 0$ otherwise, has valuation γ_0, therefore v is a surjective function. Also, if $v(f) = \gamma_1$ and $v(g) = \gamma_2$, then $\gamma_1 + \gamma_2 \in \mathrm{Supp}(fg)$ because $\delta < \gamma_1$ implies $a_\delta = 0$, $\gamma_1 < \delta$ implies $\gamma_1 + \gamma_2 - \delta < \gamma_2$ and $b_{\gamma_1+\gamma_2-\delta} = 0$, so that

$$\sum_{\delta\in\Gamma} a_\delta b_{\gamma_1+\gamma_2-\delta} = a_{\gamma_1} b_{\gamma_2} \neq 0;$$

if $\gamma < \gamma_1 + \gamma_2$ then $\gamma - \gamma_2 < \gamma_1$, so for any $\delta \in \Gamma$ either $\delta < \gamma_1$ or $\gamma - \gamma_2 < \delta$, which yield $a_\delta b_{\gamma-\delta} = 0$ for any ε, i.e. $\sum_{\delta\in\Gamma} a_\delta b_{\gamma-\delta} = 0$, so that

$$v(fg) = \min(\mathrm{Supp}(fg)) = \gamma_1 + \gamma_2 = v(f) + v(g).$$

Moreover, without loss of generality, $\gamma_1 \leq \gamma_2$, therefore

$$v(f + g) = \min(\mathrm{Supp}(f + g)) \geq \gamma_1 = \min\{v(f), v(g)\}.$$

Before showing that any nonzero element of $K(\!(t^\Gamma)\!)$ is invertible, let us prove that this ring, together with v, satisfies that any pseudo-Cauchy sequence $(f_i)_{i<\rho}$ has a pseudo-limit. For any $i < \rho$, let $f_i = \sum_\gamma a_{i,\gamma} t^\gamma$. We want to construct a formal power series $g = \sum_\gamma b_\gamma t^\gamma \in K((t^\Gamma))$ such that $v(g - f_i) = v(f_{i+1} - f_i)$ for any $i < \rho$. To that end, let $\delta_i \in \Gamma$ be the valuation $v(f_{i+1} - f_i)$. Note that for any $i < j < \rho$, we have that

$$v(f_j - f_i) = v(f_j - f_{i+1} + f_{i+1} - f_i) = v(f_{i+1} - f_i) = \delta_i,$$

because $v(f_j - f_{i+1}) > v(f_{i+1} - f_i)$ by the pseudo-Cauchy condition on $(f_i)_{i<\rho}$. Similarly, $i < j < \rho$ implies

$$\delta_j = v(f_{j+1} - f_j) > v(f_j - f_i) = v(f_{i+1} - f_i) = \delta_i.$$

Therefore, for any $i < j < \rho$ and any $\gamma < \delta_i$ we have that $a_{j,\gamma} - a_{i,\gamma} = 0$, i.e., $a_{j,\gamma} = a_{i,\gamma}$. This shows that

$$b_\gamma = \begin{cases} a_{i,\gamma} & \text{if there is some } \delta_i > \gamma, \\ 0 & \text{otherwise} \end{cases}$$

is well defined. Hence, if $i < \rho$, then $\delta_i = v(f_{i+1} - f_i) \in \mathrm{Supp}(g - f_i)$ because $b_{\delta_i} = a_{i+1,\delta_i}$ and

$$b_{\delta_i} - a_{i,\delta_i} = a_{i+1,\delta_i} - a_{i,\delta_i} \neq 0,$$

and if $\gamma < \delta_i$ then $b_\gamma = a_{i,\gamma}$ and $b_\gamma - a_{i,\gamma} = 0$. It remains to see that $g \in K(\!(t^\Gamma)\!)$, i.e., that $\mathrm{Supp}(g)$ is well ordered. Suppose otherwise, yielding an infinite descending chain $(\gamma_n)_{n<\omega} \subseteq \mathrm{Supp}(g)$. Then there is some $i < \rho$ such that $\gamma_0 < \delta_i$, for otherwise $b_{\gamma_0} = 0$. Then, for any $n < \omega$, $b_{\gamma_n} = a_{i,\gamma_n} \neq 0$ and $\gamma_n \in \mathrm{Supp}(f_i)$, which is absurd.

Now we are ready to prove that $K(\!(t^\Gamma)\!)$ is a (valued) field. Let $f = \sum_\gamma a_\gamma t^\gamma \in K(\!(t^\Gamma)\!)$ be a non-zero element. Without loss of generality, if $v(f) = \gamma$, then $a_\gamma = 1$ just by dividing f by a_γ, which we know is not zero. Let

$$\Delta = \{v(1 - fg) : g \in K(\!(t^\Gamma)\!), 1 - fg \neq 0\}.$$

If Δ has no maximal element, we can choose a cofinal sequence $(v(1-fg_i))_{i<\rho}$ of Δ, whose corresponding sequence $(g_i)_{i<\rho}$ is pseudo-Cauchy. This is because for any pair $i < j < \rho$ we have that $v(1 - fg_i) < v(1 - fg_j)$, implying that

$$v(f) + v(g_j - g_i) = v(fg_j - fg_i) = v(fg_j - 1 + 1 - fg_i) = v(1 - fg_i),$$

i.e., $v(g_j - g_i) = v(1 - fg_i) - v(f)$, and that

$$v(g_k - g_j) = v(1 - fg_j) - v(f) > v(1 - fg_i) - v(f) = v(g_j - g_i)$$

whenever $i < j < k < \rho$. Then any pseudo-limit h of $(g_i)_{i<\rho}$ must satisfy $1 - fh = 0$, for otherwise $v(1 - fh) \in \Delta$, yielding some $i < \rho$ such that $v(1 - fh) < v(1 - fg_i)$, but

$$v(fh - fg_i) = v(f) + v(h - g_i) = v(f) + v(g_{i+1} - g_i) = v(1 - fg_i) > v(1 - fh)$$

yields

$$v(1 - fg_i) = v(1 - fh + fh - fg_i) = v(1 - fh),$$

which is absurd. Therefore, it remains to prove that Δ has no maximal element. To that end, let $g \in K(\!(t^\Gamma)\!)$ be such that $v(1 - fg) < \infty$. Write $f = t^{\gamma_0} + a_{\gamma_1} t^{\gamma_1} + r_1$[5] with $a_{\gamma_1} \neq 0$, $\gamma_0 < \gamma_1$ and $v(r_1) > \gamma_1$, and write $1 - fg = b_{\delta_0} t^{\delta_0} + r_2$ with $b_{\delta_0} \neq 0$ and $v(r_2) > \delta_0$. We would like to construct some $h \in K(\!(t^\Gamma)\!)$ such that $v(1 - fg) < v(1 - fh) < \infty$. Let $h = g + g'$ where $g' = c_{\varepsilon_0} t^{\varepsilon_0} + t^{\varepsilon_1}$, $\varepsilon_1 > \varepsilon_0$, so that we can choose $\varepsilon_0, c_{\varepsilon_0}$ and ε_1 to fulfill our requirements. In this way,

$$fg' = c_{\varepsilon_0} t^{\gamma_0 + \varepsilon_0} + a_{\gamma_1} c_{\varepsilon_0} t^{\gamma_1 + \varepsilon_0} + c_{\varepsilon_0} t^{\varepsilon_0} r_1 + t^{\varepsilon_1} f.$$

Denote $R = r_2 - c_{\varepsilon_0} t^{\varepsilon_0} r_1 - t^{\varepsilon_1} f$, so that

$$\begin{aligned} 1 - fh &= 1 - fg - fg' \\ &= b_{\delta_0} t^{\delta_0} - c_{\varepsilon_0} t^{\gamma_0 + \varepsilon_0} - a_{\gamma_1} c_{\varepsilon_0} t^{\gamma_1 + \varepsilon_0} + R. \end{aligned}$$

Choose ε_0 such that $\gamma_0 + \varepsilon_0 = \delta_0$, i.e. $\varepsilon_0 = \delta_0 - \gamma_0$, and choose $c_{\varepsilon_0} = b_{\delta_0}$, in order to get

$$1 - fh = -a_{\gamma_1} b_{\delta_0} t^{\gamma_1 - \gamma_0 + \delta_0} + R.$$

Clearly $\gamma_1 - \gamma_0 + \delta_0 > \delta_0$. We now must choose some ε_1 such that $v(R) > \delta_0$. Since

$$R = r_2 - b_{\delta_0} t^{\delta_0 - \gamma_0} r_1 - t^{\varepsilon_1} f,$$

$v(r_2) > \delta_0$ and $v(-b_{\delta_0} t^{\delta_0 - \gamma_0} r_1) = \delta_0 - \gamma_0 + v(r_1) > \gamma_1 - \gamma_0 + \delta_0 > \delta_0$, we must choose ε_1 such that $v(-t^{\varepsilon_1} f) = \varepsilon_1 + v(f) > \delta_0$, i.e., letting $\varepsilon_1 > \delta_0 - v(f) = \delta_0 - \gamma_0$, and big enough so that $v(1 - fh) < \infty$.

As a summary, we have proved the following

Proposition 2.3.10. *Let K be any field and Γ be any ordered abelian group. Then $(K(\!(t^\Gamma)\!), v)$ as defined above is a pseudo-complete valued field.*

Because of Proposition 2.3.8, the Hahn series field $(K(\!(t^\Gamma)\!), v)$ is maximal and thus henselian. It is clear that its value group is Γ. The corresponding valuation ring and its maximal ideal correspond to

$$K[\![t^\Gamma]\!] = \{f \in K(\!(t^\Gamma)\!) : v(f) \geq 0\}, \ \mathscr{M} = \{f \in K[\![t^\Gamma]\!] : v(f) > 0\}$$

respectively. The function "evaluation at zero" $\varphi : K[\![t^\Gamma]\!] \longrightarrow K$ is a surjective ring homomorphism having $\mathscr{M}$ as kernel, so that the residue field of $(K(\!(t^\Gamma)\!), v)$ is isomorphic to K.

Another important consequence of maximality is that if K is algebraically closed and Γ is divisible, then $K(\!(t^\Gamma)\!)$ is algebraically closed. If it were not the case, $K(\!(t^\Gamma)\!)$ would admit a proper algebraic extension, but immediate by Proposition 2.2.1, contradicting maximality. This can be summarized as a

[5] If $f = t^\gamma$, then f is already invertible.

Corollary 2.3.11. *Let K be a field and let Γ be an ordered abelian group. Then $K(\!(t^\Gamma)\!)$ is algebraically closed if and only if K is algebraically closed and Γ is divisible.*

Now let us return to Kaplansky theory. Let (K, v) be a valued field, and let $(a_\nu)_{\nu<\rho}$ be a pseudo-Cauchy sequence in K. Then the set

$$\pi(x) = \{v(x - a_\nu) = v(a_{\nu+1} - a_\nu) : \nu < \rho\}$$

is finitely consistent, because if $\nu_1 < \ldots < \nu_n < \nu_{n+1} < \nu < \rho$, then $v(a_\nu - a_{\nu_i+1}) > v(a_{\nu_i+1} - a_{\nu_i})$ implies

$$v(a_\nu - a_{\nu_i}) = v(a_\nu - a_{\nu_i+1} + a_{\nu_i+1} - a_{\nu_i}) = v(a_{\nu_i+1} - a_{\nu_i})$$

for any $i \in \{1, \ldots, n\}$, therefore there is an elementary extension[6] (K', v') of (K, v) in which $\pi(x)$ is satisfiable, i.e., in which $(a_\nu)_{\nu<\rho}$ has a pseudo-limit $a \in K'$. Then, the (ring) type of a over K determines the type of the sequence $(a_\nu)_{\nu<\rho}$ as follows:

Definition 2.3.12. We say that the pseudo-Cauchy sequence $(a_\nu)_{\nu<\rho}$ is of *transcendental type over K* if for any non-constant polynomial $f \in K[X]$ we have that 0 is not a pseudo-limit of $(f(a_\nu))_{\nu<\rho}$, otherwise we call such a sequence *of algebraic type over K*.

If $(a_\nu)_{\nu<\rho}$ is of transcendental type over K, by Proposition 2.3.8, for any non-constant polynomial $f \in K[X]$, the sequence $\big(v(f(a_\nu))\big)_{\nu<\rho}$ is not strictly increasing, so there must be some eventual value of the latter sequence. This suggests a way to extend v to a valuation w on $K(X)$, namely the one that assigns f to the eventual value of the sequence $\big(v(f(a_\nu))\big)_{\nu<\rho}$. This is indeed a valuation, for if $f, g \in K[X]$, then there are some ordinals $\nu_1, \nu_2, \nu_3 < \rho$ such that $v(f(a_\nu)) = w(f)$ for any $\nu > \nu_1$, $v(g(a_\nu)) = w(g)$ for any $\nu > \nu_2$ and $v((f+g)(a_\nu)) = w(f+g)$ for any $\nu > \nu_3$, so that

$$v(fg(a_\nu)) = v(f(a_\nu)) + v(g(a_\nu)) = w(f) + w(g)$$

for any $\nu > \max\{\nu_1, \nu_2\}$ and

$$w(f+g) = v((f+g)(a_\nu)) \geq \min\{v(f(a_\nu), v(g(a_\nu)))\} = \min\{w(f), w(g)\}$$

for any $\nu > \max\{\nu_1, \nu_2, \nu_3\}$.

Moreover, $X \in K[X]$ is a pseudo-limit of the original pseudo-Cauchy sequence $(a_\nu)_{\nu<\rho}$, because for any $\nu_0 < \rho$, $w(X - a_{\nu_0})$ is the eventual value of $(v(a_\nu - a_{\nu_0}))_{\nu<\rho}$, which equals $v(a_{\nu_0+1} - a_{\nu_0})$.

Clearly $\Gamma_v = \Gamma_w$. In fact, the extension $(K(X), w) \supseteq (K, v)$ is immediate. Before proving this, let us prove an auxiliary lemma concerning "pseudo-continuity" of polynomials.

Lemma 2.3.13. *Let $(a_\nu)_{\nu<\rho}$ a pseudo-Cauchy sequence in a valued field (K, v), and let $f \in K[X]$. Then $(f(a_\nu))_{\nu<\rho}$ is pseudo-Cauchy and $f(a)$ is a pseudo-limit of $(f(a_\nu))_{\nu<\rho}$ whenever a is a pseudo-limit of $(a_\nu)_{\nu<\rho}$.*

Proof. Let a be a pseudo-limit of the sequence $(a_\nu)_{\nu<\rho}$ in an extension of (K, v). Then, by Lemma 2.3.1, we can write

$$f(a_\nu) - f(a) = \sum_{i=1}^{n} f_i(a)(a_\nu - a)^i.$$

Since $v(f_i(a)(a_\nu - a)^i) = v(f_i) + iv(a_\nu - a)$ and $v(a_\nu - a)$ is strictly increasing, there is an index $i_0 \in \{1, \ldots, n\}$ such that $v(f_{i_0}) + i_0 v(a_\nu - a) < v(f_i) + iv(a_\nu - a)$ for any $i \neq i_0$ eventually. $\square$

[6] In a two sorted language corresponding to K and $v(K)$, or in $\mathscr{L}_{div}$ (see below).

Now, let $f \in K[X]$ be such that $w(f) = 0$, i.e., such that there is some $\nu_0 < \rho$ satisfying $v(f(a_\nu)) = w(f)$ for any $\nu > \nu_0$. By pseudo-continuity of f, $f(X) = f$ is a w-pseudo-limit of $(f(a_\nu))_{\nu<\rho}$, i.e. the sequence $\big(w(f - f(a_\nu))\big)_{\nu<\rho}$ is eventually strictly increasing. This implies that there must be some ν such that $w(f - f(a_\nu)) > 0$, this is, $f + \mathscr{M}_w = f(a_\nu) + \mathscr{M}_w$. This result extends to arbitrary fractions $\frac{f}{g} \in K(X)$. If $w(\frac{f}{g}) = 0$ then $w(f) = w(g) \in \Gamma_v$, i.e. there is some $a \in K^\times$ such that $w(f) = w(g) = v(a)$. Then we can run the argument above with the polynomials $\frac{f}{a}$ and $\frac{g}{a}$, yielding a pair of ordinal numbers $\nu_1, \nu_2 < \rho$ such that $\frac{f}{a} + \mathscr{M}_w = f(a_{\nu_1}) + \mathscr{M}_w$ and $\frac{g}{a} + \mathscr{M}_w = g(a_{\nu_2}) + \mathscr{M}_w$. Therefore

$$\frac{f}{g} + \mathscr{M}_w = \frac{f(a_{\nu_1})}{g(a_{\nu_2})} + \mathscr{M}_w$$

as we wanted. We have proved the following

Proposition 2.3.14. *Let $s = (a_\nu)_{\nu<\rho}$ be a pseudo-Cauchy sequence in (K, v) without pseudo-limit in K. If s is of transcendental type over K, then there is a proper immediate extension $(K(X), w)$ of (K, v) in which X is a pseudo-limit of s.*

The same is true for sequences of algebraic type over K. We will state the corresponding result without proof, and refer the reader to [12].

Fact 2.3.15. *Let $s = (a_\nu)_{\nu<\rho}$ be a pseudo-Cauchy sequence in (K, v) without pseudo-limit in K. If s is of algebraic type over K and $\mu \in K[X]$ is a non-constant polynomial of minimal degree such that $(\mu(a_\nu))_{\nu<\rho}$ has 0 as pseudo-limit, then*

1. *μ is irreducible and $\deg(\mu) \geq 2$,*
2. *there is an immediate proper extension $(K(a), w)$ of (K, v) such that a is a pseudo-limit of $(a_\nu)_{\nu<\rho}$ and $\mu(a) = 0$.*

As a summary, whenever (K, v) is not pseudo-complete, then it has some proper immediate extension, i.e., (K, v) is not maximal. This is,

Corollary 2.3.16. *Let (K, v) be a valued field. Then (K, v) is pseudo-complete if and only if it is maximal.*

Fact 2.3.15 implies that whenever (K, v) has a pseudo-Cauchy sequence of algebraic type over K without pseudo-limit in K, then there is an algebraic proper immediate extension $(K(a), w)$ of (K, v). We then say that a valued field is *algebraically maximal* if it does not admit any algebraic proper immediate extension. Then the statement above can be restated as saying that any pseudo-Cauchy sequence in an algebraically maximal valued field (K, v) must have a pseudo-limit in K.

The converse of the latter statement is, in fact, also true, because if $(L, w) \supseteq (K, v)$ is an algebraic proper immediate extension and $z \in L \setminus K$, then by Proposition 2.3.8, there is a pseudo-Cauchy sequence $(a_\nu)_{\nu<\rho}$ in K having z as a pseudo-limit and having none in K. If $\mu \in K[X]$ is the minimal polynomial of z over K, it follows from pseudo-continuity that $(\mu(a_\nu))_{\nu<\rho}$ has $\mu(z) = 0$ as a pseudo-limit, i.e., $(a_\nu)_{\nu<\rho}$ is of algebraic type over K. This is,

Corollary 2.3.17. *Let (K, v) be a valued field. Then it is algebraically maximal if and only if any pseudo-Cauchy sequence in K of algebraic type over K has a pseudo-limit in K.*

As a consequence of the fundamental inequality, if (K, v) is henselian of residue characteristic 0, then for any finite extension (L, w) we have that

$$[L : K] = [k_w : k_v][\Gamma_w : \Gamma_v],$$

so (L, w) cannot be a proper immediate extension. Together with the paragraph after Proposition 2.3.8, we have proved the following

Corollary 2.3.18. *Let (K, v) be a valued field of residue characteristic 0. Then it is henselian if and only if it is algebraically maximal. In particular, its henselization K^h is the unique algebraically maximal extension of (K, v).*

2.3.2 Pas' Theorem

We now discuss the model theory of valued fields. First of all, the language in which we will be mainly interested in is the *Denef-Pas* language $\mathscr{L}_{DP}$, which is the following three-sorted language.

The first sort **VF** has the language of rings, and is meant to be interpreted by a valued field K. The second sort $\mathbf{\Gamma}$ has the language of ordered abelian groups with the symbol ∞, which will be intepreted by the value group Γ_v of the valued field of the first sort, in the usual way. The third sort **k** has the language of rings and is meant to be interpreted by the residue class field k_v of the valued field of the first sort. We will equip these sorts with a symbol $v : \mathbf{VF} \longrightarrow \mathbf{\Gamma}$ to be interpreted as the valuation v, and a symbol $\mathrm{ac} : \mathbf{VF} \longrightarrow \mathbf{k}$ to be interpreted as an *angular component map,* i.e. a surjective function $\mathrm{ac} : K \longrightarrow k_v$ such that

1. $\mathrm{ac}(x) = 0$ if and only if $x = 0$,
2. $\mathrm{ac} \restriction_{K^\times} : K^\times \longrightarrow k_v^\times$ is a group homomorphism, and
3. $\mathrm{ac}(x) = x + \mathscr{M}_v$ for any $x \in \mathcal{O}_v^\times$.

Note that the residue map is interpretable in this language, by defining it as ac in any element whose valuation is zero, and as zero otherwise.

If K is any field and Γ is any ordered abelian group, then the function $\mathrm{ac} : K(\!(t^\Gamma)\!) \longrightarrow K$ defined as $\mathrm{ac}(f) = a_{v(f)}$ whenever $f = \sum_\gamma a_\gamma t^\gamma$ defines an angular component map.

The theory of henselian valued fields of equicharacteristic 0 admitting an angular component map is denoted by T_{Pas}. We now state the main result of this section, due to Pas.

Let Δ_0 be the following set of $\mathscr{L}_{DP}$-formulas:

- ✓ $\varphi(\overline{x})$, where $\overline{x}$ is a tuple of variables of the valued field sort and $\varphi(\overline{x})$ is a *quantifier free* $\mathscr{L}_{ring}$-formula,
- ✓ $\psi(v(t_1(\overline{x})), \dots, v(t_n(\overline{x})), \overline{y})$ where the t_i are $\mathscr{L}_{ring}$-terms, $\overline{x}$ is a tuple of variables of the valued field sort, $\overline{y}$ is a tuple of variables of the value group and $\psi(y_1, \dots, y_n, \overline{y})$ is any $\mathscr{L}_{oag}$-formula,
- ✓ $\theta(\mathrm{ac}(t_1(\overline{x})), \dots, \mathrm{ac}(t_n(\overline{x})), \overline{z})$ where the t_i are $\mathscr{L}_{ring}$-terms, $\overline{x}$ is a tuple of variables of the valued field sort, $\overline{z}$ is a tuple of variables of the residue field and $\theta(z_1, \dots, z_n, \overline{z})$ is any $\mathscr{L}_{ring}$-formula.

Let Δ be the set of boolean combinations of formulas from Δ_0.

Theorem 2.3.19 (Pas' Theorem, cf. Theorem 6.7 of [9]). *The theory T_{Pas} eliminates* **VF**-*quantifiers, i.e. any $\mathscr{L}_{DP}$-formula is equivalent to some Δ-formula, modulo T.*

Before proving this theorem, we need a couple of lemmas.

Lemma 2.3.20. *Let (K, v, ac) be a valued field with angular component.*

1. *If $x \in K^\times$ and $y \in K^\times$ satisfies $v(y-x) > v(x)$, then $\mathrm{ac}(x) = \mathrm{ac}(y)$.*
2. *If $(L, w) \supseteq (K, v)$ satisfies $\Gamma_w = \Gamma_v$, then ac extends uniquely to an ac-map in L.*
3. *If $(L, w, \mathrm{ac}_w) \supseteq (K, v, \mathrm{ac}_v)$, then the extension ac_w is determined by its values on representatives of generators of the group Γ_w/Γ_v.*

Proof. 1. If $v(y-x) > v(x)$, then $v(\frac{y}{x} - 1) > 0$ and $v(y) = v(y - x + x) = v(x)$, so $v(\frac{y}{x}) = 0$. Therefore $\mathrm{ac}(\frac{y}{x}) = \frac{y}{x} + \mathscr{M}_v = 1 + \mathscr{M}_v$, so $\mathrm{ac}(y) = \mathrm{ac}(x)$.

2. Suppose ac_w is an ac-map in L that extends ac. Since $\Gamma_w = \Gamma_v$, for any $y \in L^\times$ there is some $x \in K^\times$ such that $w(y) = v(x) = w(x)$, so that $w(\frac{y}{x}) = 0$. Then $\mathrm{ac}_w(\frac{y}{x}) = \mathrm{res}(\frac{y}{x})$, i.e., $\mathrm{ac}_w(y) = \mathrm{ac}_w(x)\,\mathrm{res}(\frac{y}{x}) = \mathrm{ac}_v(x)\,\mathrm{res}(\frac{y}{x})$. This is in turn a well defined function.

3. Suppose $S \subseteq \Gamma_w/\Gamma_v$ is a set of generators, and let $S^* = \{a \in L^\times : w(a) + \Gamma_v \in S\}$. Suppose ac_w and ac'_w are two ac-maps in L, extending ac_v, that coincide in S^*, and let $y \in L^\times$. Then $w(y) + \Gamma_v = \sum_i w(a_i) + \Gamma_v = w(a_1 \dots a_n) + \Gamma_v$ for some $a_1, \dots, a_n \in S^*$, so $w(\frac{y}{a_1 \dots a_n}) = w(x)$ for some $x \in K^\times$. Therefore $w(\frac{y}{xa_1 \dots a_n}) = 0$ and $\mathrm{ac}_w(\frac{y}{xa_1 \dots a_n}) = \mathrm{res}(\frac{y}{xa_1 \dots a_n})$. Analogously $\mathrm{ac}'_w(\frac{y}{xa_1 \dots a_n}) = \mathrm{res}(\frac{y}{xa_1 \dots a_n})$, so

$$\begin{aligned} \mathrm{ac}_w(y) &= \mathrm{ac}_v(x)\,\mathrm{ac}_w(a_1 \dots a_n)\,\mathrm{res}\left(\frac{y}{xa_1 \dots a_n}\right) \\ &= \mathrm{ac}_v(x)\,\mathrm{ac}'_w(a_1 \dots a_n)\,\mathrm{res}\left(\frac{y}{xa_1 \dots a_n}\right) \\ &= \mathrm{ac}'_w(y). \end{aligned}$$

$\square$

Lemma 2.3.21. *Let $(L, w) \supseteq (K, v)$ be an extension of valued fields with residue characteristic 0 such that (L, w) is henselian and $k_w = k_v$.*

1. *For any $a \in \mathscr{M}_w$ and any positive integer n, there is some $b \in \mathcal{O}_w$ such that $b^n = 1 + a$.*
2. *Let $\gamma \in \Gamma_w \cap \mathrm{Div}(\Gamma_v)$ and let n be minimal such that $n\gamma \in \Gamma_v$. Then there is some $b \in L$ such that $w(b) = \gamma$ and $b^n \in K$.*

Proof. 1. Consider the polynomial $X^n - (a+1) \in \mathcal{O}_w[X]$. Then $X^n - (\overline{a+1})$ has $\overline{1}$ as a simple root, because its derivative is nX^{n-1} and $n\overline{1} \neq \overline{0}$. By Theorem 2.3.5, there is some $b \in \mathcal{O}_w$ such that $b^n - (a+1) = 0$ and $b + \mathscr{M}_w = 1 + \mathscr{M}_w$.

2. Let $a \in L^\times$ be such that $\gamma = w(a)$ and let $c \in K^\times$ be such that $nw(a) = w(c)$, i.e. $w(\frac{a^n}{c}) = 0$. Use the equality $k_w = k_v$ to find some $d \in \mathcal{O}_w \cap K$ such that $\mathrm{res}(\frac{a^n}{c}) = \mathrm{res}(d) \neq \overline{0}$, so $d \in \mathcal{O}_w^\times$ and $\mathrm{res}(\frac{a^n}{cd}) = \overline{1}$. Then $\frac{a^n}{cd} - 1 = e \in \mathscr{M}_w$, so that item 1 above implies that $\frac{a^n}{cd} = e + 1 = f^n$ for some $f \in \mathcal{O}_w$. Thus $cd = (\frac{a}{f})^n \in K$ and $w(f^n) = w(1+e) = 0$, so $w(f) = 0$ and $w(\frac{a}{f}) = w(a) = \gamma$.

$\square$

Let $\mathfrak{M} = (K, \Gamma_v, k_v), \mathfrak{N} = (L, \Gamma_w, k_w)$ be models of T_{Pas}, with K being countable and $\mathfrak{N}$ being $\aleph_1$-saturated. Let $A = (\mathbf{VF}(A), \mathbf{\Gamma}(A), \mathbf{k}(A))$ be a substructure of $\mathfrak{M}$ and let $f = (f_{\mathbf{VF}}, f_{\mathbf{\Gamma}}, f_{\mathbf{k}}) : A \longrightarrow \mathfrak{N}$ be an $\mathscr{L}_{DP}$-embedding such that $f_{\mathbf{\Gamma}}$ and $f_{\mathbf{k}}$ are elementary. We aim to extend f to an $\mathscr{L}_{DP}$-embedding $\widetilde{f} = (\widetilde{f_{\mathbf{VF}}}, \widetilde{f_{\mathbf{\Gamma}}}, \widetilde{f_{\mathbf{k}}}) : \mathfrak{M} \longrightarrow \mathfrak{N}$.

If $\gamma \in \Gamma_v \setminus \mathbf{\Gamma}(A)$, then the pushforward of $\mathrm{tp}(\gamma/\mathbf{\Gamma}(A))$ via $f_{\mathbf{\Gamma}}$ is a type in $\mathfrak{N}$ with a countable set of parameters, realised by some $\gamma^* \in \Gamma_w$ because of $\aleph_1$-saturation. Then $\widetilde{f_{\mathbf{\Gamma}}}$ can be defined in γ as γ^*, which is again elementary. In the same way it is possible to define $\widetilde{f_{\mathbf{k}}}$ in any element of $k_v \setminus \mathbf{k}(A)$.

It is also possible to extend $f_{\mathbf{VF}}$ to the field generated by $\mathbf{VF}(A)$, by defining $\widetilde{f_{\mathbf{VF}}}(a^{-1}) = (f_{\mathbf{VF}}(a))^{-1}$. In this way, we define $v(a^{-1}) = -v(a)$ and $\mathrm{ac}(a^{-1}) = (ac(a))^{-1}$ for any non-zero element $a \in \mathbf{VF}(A)$

Thus we may assume that $A = (\mathbf{VF}(A), \Gamma_v, k_v)$, where $\mathbf{VF}(A)$ is a valued field with valuation $v \restriction_{\mathbf{VF}(A)}$. Let $\Gamma_{\mathbf{VF}}$ and $k_{\mathbf{VF}} = {\mathcal{O}_{\mathbf{VF}}}/{\mathscr{M}_{\mathbf{VF}}}$ be the corresponding value group and residue class field, so that $\Gamma_{\mathbf{VF}}$ is a subgroup of Γ_v and $k_{\mathbf{VF}}$ is a subfield of k_v. First, we seek to extend $f_{\mathbf{VF}}$ so that we may assume that $k_{\mathbf{VF}} = k_v$.

1. If $\alpha \in k_v \cap k_{\mathbf{VF}}^{alg}$, let $\overline{f_{min}} \in k_{\mathbf{VF}}[X]$ be its minimal polynomial. Then α is a simple root of $\overline{f_{min}}$, so by henselianity of $(\mathbf{VF}(A)^{alg} \cap K, v \restriction_{\mathbf{VF}(A)^{alg} \cap K})$, there is some root $a \in \mathcal{O}_{\mathbf{VF}(A)^{alg}}$ of a lift $f_{min} \in \mathcal{O}_{\mathbf{VF}}[X]$ of $\overline{f_{min}}$ satisfying $a + \mathscr{M}_v = \alpha$. This can be achieved since $char(K) = 0$ implies that relative algebraic closure equals relative separable closure, so henselianity of $(\mathbf{VF}(A)^{alg} \cap K, v \restriction_{\mathbf{VF}(A)^{alg} \cap K})$ follows from Corollary 2.3.6. Let $b \in L$ be a root of $f_{\mathbf{VF}}(f_{min})$ such that $b + m_w = \widetilde{f_{\mathbf{k}}}(\alpha)$. By the fundamental inequality, $v \restriction_{\mathbf{VF}(A)}$ extends uniquely to $\mathbf{VF}(A)(a)$ whenever such extension has $k_{\mathbf{VF}}(\alpha)$ as residue field. By the same reason, such an extension satisfies that $[\Gamma_{\mathbf{VF}(A)(a)} : \Gamma_{\mathbf{VF}(A)}] = 1$, therefore ac $\restriction_{\mathbf{VF}}$ also extends uniquely to this field by Lemma 2.3.20. Then we can define $\widetilde{f_{\mathbf{VF}}}(a) = b$.

2. If $\alpha \in k_v \setminus k_{\mathbf{VF}}^{alg}$ and $a + \mathscr{M}_v = \alpha$ for some $a \in K^\times$, then $a \notin \mathcal{O}_{\mathbf{VF}}^{alg}$, for otherwise there would be some $g \in \mathcal{O}_{\mathbf{VF}}[X]$ such that $g(a) = 0$, so $\overline{g} \in k_{\mathbf{VF}}[X]$ and $\overline{g}(\overline{a}) = \overline{g}(\alpha) = 0$. By Observation 2.3.4, a is transcendental over $\mathbf{VF}(A)$. Let $b \in L$ be such that b is not algebraic over $f_{\mathbf{VF}}(\mathcal{O}_{\mathbf{VF}})$ and $b + \mathscr{M}_w = \widetilde{f_{\mathbf{k}}}(\alpha)$. The Gauss extension of $v \restriction_{\mathbf{VF}}$ to $\mathbf{VF}(A)(a)$ is the unique possible extension having $k_{\mathbf{VF}}(\alpha)$ as residue field and not extending the value group, as stated in Lemma 3.19 of [18]. Again, by Lemma 2.3.20, ac $\restriction_{\mathbf{VF}}$ extends uniquely to $\mathbf{VF}(A)(a)$. Then we can define $\widetilde{f_{\mathbf{VF}}}(a) = b$.

Now we deal with value groups, and seek to extend $\widetilde{f_{\mathbf{VF}}}$ so that we can also assume that $\Gamma_v = \Gamma_{\mathbf{VF}}$.

3. If $\gamma \in \Gamma_v \cap \mathrm{Div}(\Gamma_{\mathbf{VF}})$, use Lemma 2.3.21 to find some $a \in K$ such that $v(a) = \gamma$ and $a^n \in \mathbf{VF}(A)$, where n is the least positive integer satisfying $n\gamma \in \Gamma_{\mathbf{VF}}$. By the fundamental inequality, there is a unique extension of $v \restriction_{\mathbf{VF}}$ to $\mathbf{VF}(A)(a)$ having $\Gamma_{\mathbf{VF}}(\gamma)$ as value group, and whose residue field does not grow. Now we want to find some $b \in L$ such that $w(b) = \widetilde{f_{\mathbf{\Gamma}}}(\gamma)$ and $\mathrm{ac}_w(b) = \widetilde{f_{\mathbf{k}}}(\mathrm{ac}(a))$, in order to define $\widetilde{f_{\mathbf{VF}}}(a) = b$. Let $c \in L$ be such that $w(c) = \widetilde{f_{\mathbf{\Gamma}}}(\gamma)$ and $c^n \in f_{\mathbf{VF}}(\mathbf{VF}(A))$. Therefore $\mathrm{ac}(c)^n \in k_{f_{\mathbf{VF}}(\mathbf{VF}(A))} = \widetilde{f_{\mathbf{k}}}(k_v)$. Since $\widetilde{f_{\mathbf{k}}}(k_v) \preceq k_w$, the subfield $\widetilde{f_{\mathbf{k}}}(k_v)$ is relatively algebraically closed in k_w, so $\mathrm{ac}(c) \in \widetilde{f_{\mathbf{k}}}(k_v)$. Therefore $\widetilde{f_{\mathbf{k}}}(\mathrm{ac}(a))\,\mathrm{ac}(c^{-1})$ is also an element of $\widetilde{f_{\mathbf{k}}}(k_v)$, so there exists some $d \in \mathcal{O}^\times_{f_{\mathbf{VF}}(\mathbf{VF}(A))}$ such that $\mathrm{res}(d) = \widetilde{f_{\mathbf{k}}}(\mathrm{ac}(a))\,\mathrm{ac}(c^{-1})$. Then $\widetilde{f_{\mathbf{k}}}(\mathrm{ac}(a)) = \mathrm{ac}(cd)$ and $w(cd) = \widetilde{f_{\mathbf{\Gamma}}}(\gamma)$, so $b = cd$ is as wanted.

4. If $\gamma \in \Gamma_v \setminus \mathrm{Div}(\Gamma_{\mathbf{VF}})$, let $a \in K$ be such that $v(a) = \gamma$. Then $\mathrm{ac}(a) \in k_v = k_{\mathbf{VF}}$, so there is some $c \in \mathcal{O}_{\mathbf{VF}}^{\times}$ such that $\mathrm{ac}(a) = \mathrm{ac}(c)$, so $\mathrm{ac}(\frac{a}{c}) = 1$. Find some $b \in L$ satisfying $w(b) = \widetilde{f_{\mathbf{\Gamma}}}(\gamma)$ and $\mathrm{ac}(b) = 1$. Since $\gamma \notin \mathrm{Div}(\Gamma_{\mathbf{VF}})$, then a is transcendental over $\mathbf{VF}(A)$, as stated in Lemma 3.20 of [18]. Therefore the extension given by $\widetilde{f_{\mathbf{VF}}}(a) = b$ is uniquely determined by the Gauss extension of $v\restriction_{\mathbf{VF}(A)}$, which in this case satisfies that its value group is $\Gamma_{\mathbf{VF}} \oplus \mathbb{Z}\gamma$ and whose residue field does not grow.

Finally, we arrive at assuming that the extension $K/\mathbf{VF}(A)$ is immediate. Since henselizations are immediate extensions, we may assume also that $\mathbf{VF}(A)$ is henselian.

5. Let $a \in K \setminus \mathbf{VF}(A)$. By Lemma 2.3.8, there is a pseudo-Cauchy sequence $(a_\nu)_{\nu<\omega}$ in $\mathbf{VF}(A)$ having a as pseudo-limit in K but with none in $\mathbf{VF}(A)$. By Corollary 2.3.18, $\mathbf{VF}(A)$ is algebraically maximal, so that $(a_\nu)_{\nu<\omega}$ is of transcendental type over $\mathbf{VF}(A)$. Therefore, by Proposition 2.3.14, the extension $K/\mathbf{VF}(A)$ is determined by a being a pseudo-limit of $(a_\nu)_{\nu<\omega}$, so we can find some $b \in L$ such that b is a pseudo-limit of $(f_{\mathbf{VF}}(a_\nu))_{\nu<\omega}$. Then we can define $\widetilde{f_{\mathbf{VF}}}(a) = b$.

3 The Dp-finite Case

3.1 Dp-finite Hahn Series Fields

In this section we will present a construction of dp-finite Hahn series fields. First, we will use the following result as a black box. It can be found in [17], Theorem 2.2.7, where it refers to burden instead of dp-rank. These two notions coincide whenever the theory is dependent, and can be computed via inp-patterns as in Proposition 2.1.11.

Fact 3.1.1 (Cf. Theorem 2.2.7 of [17]). *Suppose T is a theory of henselian valued fields in the language $\mathscr{L}_{DP}$ admitting* **VF**-*quantifier elimination. Let T_{VG} and T_{RF} be the induced theories on the value group and the residue field respectively. Then*

$$\text{dp-rk}(T) = \text{dp-rk}(T_{RF}) + \text{dp-rk}(T_{VG}).$$

In particular, Pas' theorem guarantees that for any field K of characteristic zero and any ordered abelian group Γ,

$$\text{dp-rk}_{\mathscr{L}_{DP}}\left(K(\!(t^{\Gamma})\!)\right) = \text{dp-rk}_{\mathscr{L}_{ring}}(K) + \text{dp-rk}_{\mathscr{L}_{oag}}(\Gamma). \tag{3.1.1}$$

For positive equicharacteristic, Theorem 4.4 of [2] states that algebraically maximal *Kaplansky fields* also admit **VF**-quantifier elimination, so equation 3.1.1 also holds. The following definition can be found in section 13.11 of [13].

Definition 3.1.2. A *Kaplansky field* is a valued field of characteristic $p \geq 0$ whose value group is p-divisible whenever $p > 0$, and whose residue field is perfect and does not admit any finite separable extension of degree divisible by p.

Examples of algebraically maximal Kaplansky fields are $\mathbb{F}_p^{alg}(\!(t^{\Gamma})\!)$ for any p-divisible ordered abelian group Γ.

In order to exploit equation 3.1.1, we would like to know which ordered abelian groups and which fields are dp-finite. We begin with the case of ordered abelian groups. The following result is Theorem 1.2 of [4], and such results of classification of dp-finite ordered abelian groups have also been obtained independently by Halevi and Hasson in [7] and by Farré in [6].

Definition 3.1.3. Let $(\Gamma, +, 0, <)$ be an ordered abelian group.

1. A number $n \in \mathbb{N}$ is called *singular* for Γ if $[\Gamma : n\Gamma] = \infty$. The set of all singular primes for Γ is denoted by $\mathbb{P}_{sing}$.

2. Let n be a positive integer and let $\gamma \in \Gamma$. If $\gamma \in n\Gamma$, define $H_n(\gamma) = \{0\}$. Otherwise, let $H_n(\gamma)$ be the largest convex subgroup of Γ satisfying $\gamma \notin H_n(\gamma) + n\Gamma$.

3. For a pair $\gamma, \gamma' \in \Gamma$ we say that $\gamma \sim_n \gamma'$ if $H_n(\gamma) = H_n(\gamma')$. Define S_n to be the imaginary sort ${}^{\Gamma}\!/_{\sim_n}$.

If $[\Gamma : n\Gamma]$ is infinite and $n = p_1^{k_1} \dots p_m^{k_m}$ is the prime decomposition of n, by the Chinese Reminder Theorem there is some $p_i^{k_i}$ such that $[\Gamma : p_i^{k_i}\Gamma]$ is infinite. If $[\Gamma : p_i\Gamma]$ is finite, and since

$$\Gamma/_{p_i\Gamma} \cong {}^{\Gamma/p_i^{k_i}\Gamma}\!/_{p_i\Gamma/p_i^{k_i}\Gamma} \qquad \text{and} \qquad {}^{p_i\Gamma}\!/_{p_i^{k_i}\Gamma} \cong {}^{\Gamma}\!/_{p_i^{k_i-1}\Gamma},$$

Lagrange's theorem and an induction hypothesis would imply that

$$\begin{aligned}|\Gamma/p_i^{k_i}\Gamma| &= [\Gamma/p_i^{k_i}\Gamma : p_i\Gamma/p_i^{k_i}\Gamma]|p_i\Gamma/p_i^{k_i}\Gamma| \\ &= |{}^{\Gamma}\!/_{p_i\Gamma}||{}^{\Gamma}\!/_{p_i^{k_i-1}\Gamma}|\end{aligned}$$

is finite, which is absurd. Therefore, if Γ has no singular prime numbers, then Γ has no singular numbers at all.

Fact 3.1.4. *(Cf. Theorem 1.2 of [4]) An ordered abelian group* $(\Gamma, +, 0, <)$ *has finite dp-rank if and only if the following two conditions hold:*

- ✓ Γ *has only finitely many singular primes.*
- ✓ *For every singular prime* p*, the sort* S_p *is finite.*

Moreover, the dp-rank of Γ *in this language is bounded above by*

$$1 + \sum_{p \in \mathbb{P}_{sing}} |S_p|.$$

There is an important set of ordered abelian groups whose dp-ranks range over all finite cardinals. In the spirit of the latter theorem, they are groups with just one singular prime p, and whose sort S_p is of fixed size, or alternatively, archimedian groups whose set of singular primes is fixed. This example can be found in chapter 5 of [4].

Example 3.1.5. Let p be a prime number and let $\mathbb{Z}_{(p)}$ be the localization of the ring $\mathbb{Z}$ to the ideal generated by p. Fix a countable linearly independent set C of $\mathbb{R}$ over $\mathbb{Q}$ consisting only of irrational numbers, and let

$$G_p = \mathbb{Z}_{(p)}C = \left\{ \sum_{i=1}^{n} a_i c_i : n \in \mathbb{N}, a_i \in \mathbb{Z}_{(p)}, c_i \in C \right\},$$

ordered with the usual order in $\mathbb{R}$. Therefore, as an ordered abelian group, G_p is archimedian. If q is any prime number, then

$$\begin{aligned} qG_p &= (q\mathbb{Z}_{(p)})C \\ &= \begin{cases} \mathbb{Z}_{(p)}C & \text{if } q \neq p, \\ (p\mathbb{Z}_{(p)})C & \text{otherwise} \end{cases} \\ &= \begin{cases} G_p & \text{if } q \neq p, \\ (p\mathbb{Z}_{(p)})C & \text{otherwise,} \end{cases} \end{aligned}$$

so that

$$G_p\!/_{qG_p} \cong \begin{cases} \{0\} & \text{if } q \neq p, \\ \bigoplus_{i<\omega} \mathbb{Z}/p\mathbb{Z} & \text{otherwise,} \end{cases}$$

i.e. p is the only singular prime for G_p. Since $G_p \leq \mathbb{R}$, it cannot have non-trivial convex subgroups, so $|S_p| = 1$.

- ✓ In order to get a group of fixed rank k, we can look at

$$\mathcal{G}_{p,k} := \bigoplus_{i=1}^{k} G_p,$$

ordered lexicographically. Then p is again the only singular prime for $\mathcal{G}_{p,k}$, but now there are k convex subgroups out of which $|S_p| = k$. Together with Fact 3.1.4, the dp-rank of $\mathcal{G}_{p,k}$ is at most $1+k$. Indeed, its dp-rank *equals* $1+k$. Dolich and Goodrick exhibit an inp-pattern of length $1+k$ after the construction of $\mathcal{G}_{p,k}$ in [4].

✓ If we want to build an *archimedian* ordered abelian group with dp-rank $1+n$ for some fix $n \in \mathbb{N}$, fix a set $P = \{p_1, \ldots, p_n\}$ of prime numbers. If $b_1, \ldots, b_n \in \mathbb{R}$ are linearly independent irrationals over $\mathbb{Q}(C)$,[7] then

$$\mathcal{G}_P := \mathbb{Q} + b_1 G_{p_1} + \ldots + b_n G_{p_n} \cong \mathbb{Q} \oplus \bigoplus_{i=1}^{n} G_{p_i}$$

ordered as a substructure of $\mathbb{R}$ yields an archimedian ordered abelian group, so that if p is singular for $\mathcal{G}_P$, then $|S_p| = 1$. If q is any prime number, then

$$\begin{aligned} \mathcal{G}_P \big/ q\mathcal{G}_P &\cong \{0\} \oplus \bigoplus_{i=1}^{n} G_{p_i} \big/ qG_{p_i} \\ &\cong \begin{cases} \{0\} & \text{if } q \notin P, \\ G_{p_i} \big/ p_i G_{p_i} & \text{if } q = p_i \in P, \end{cases} \end{aligned}$$

so P is exactly $\mathbb{P}_{sing}$ for $\mathcal{G}_P$. Again, by Fact 3.1.4, the dp-rank of $\mathcal{G}_P$ is at most $1+n$.

Note that $\mathcal{G}_P$ is dense in $\mathbb{R}$ because it contains $\mathbb{Q}$, but it is not divisible. Indeed, if $\frac{b_1 c_1}{p_1}$ were an element of $\mathcal{G}_P$, by linear independence of $\{b_1, \ldots, b_n\}$ over $\mathbb{Q}(C)$ we would have that $\frac{c_1}{p_1} \in G_{p_1}$, and by linear independence of C over $\mathbb{Q}$ we would have that $\frac{1}{p_1} \in \mathbb{Z}_{(p_i)}$, which is absurd. Note also that $\mathcal{G}_P$ is p-divisible for any prime $p \notin P$. We can prove even more, namely

Proposition 3.1.6. *The dp-rank of $\mathcal{G}_P$ is exactly $1+n$.*

Proof. We already know that its dp-rank is at most $1+n$. We will present an inp-pattern for $\Gamma = \mathcal{G}_P$ of length $1+n$, in order to conclude that

$$\text{dp-rk}_{\mathscr{L}_{oag}}(\mathcal{G}_P) = 1+n.$$

The formulas will be $\varphi_0(x; y, z) : \; y < x < z$, and $\varphi_\alpha(x; y) : \; x - y \in p_\alpha \Gamma$ for any $\alpha \in \{1, \ldots, n\}$.

Define $(a_i^0)_{i<\omega} = (c_i, d_i)_{i<\omega}$ such that $c_i < d_i < c_{i+1}$ for any $i < \omega$, so the first row

$$\{\varphi_0(x; a_i^0) : i < \omega\}$$

is 2-inconsistent. Let $q = p_1 \ldots p_n$, $q_\alpha = p_1 \ldots p_{\alpha-1} p_{\alpha+1} \ldots p_n$ and let $(b_i^\alpha)_{i<\omega}$ be any infinite sequence of representatives of distinct classes of $\Gamma \big/ p_\alpha \Gamma$. These classes exist because P is exactly the set of singular primes of Γ. Now define $(a_i^\alpha)_{i<\omega} = (q_\alpha b_i^\alpha)_{i<\omega}$. Then each of the rows

$$\{\varphi_\alpha(x; a_i^\alpha) : i < \omega\}$$

is 2-inconsistent, because if $i \neq j$ and $q_\alpha(b_i^\alpha - b_j^\alpha) = p_\alpha \gamma$ for some $\gamma \in \Gamma$, then, as q_α is invertible modulo p_α, there are some $k, l \in \mathbb{Z}$ such that $p_\alpha k + 1 = q_\alpha l$, yielding $b_i^\alpha - b_j^\alpha = p_\alpha(l\gamma - k(b_i^\alpha - b_j^\alpha))$ and contradicting the choice of the sequence $(b_i^\alpha)_{i<\omega}$.

[7] These elements exist because $[\mathbb{R} : \mathbb{Q}] = [\mathbb{R} : \mathbb{Q}(C)][\mathbb{Q}(C) : \mathbb{Q}]$ is uncountable and $[\mathbb{Q}(C) : \mathbb{Q}]$ is countable.

Let $\eta : \{0, \ldots, n\} \longrightarrow \omega$ be any function. If $\gamma \in \Gamma$ then, by construction, any element γ_η of the form

$$q\gamma + \sum_{\alpha=1}^{n} a^{\alpha}_{\eta(\alpha)}$$

satisfies $\varphi_\alpha(x; a^{\alpha}_{\eta(\alpha)})$ for each $\alpha \in \{1, \ldots, n\}$, so for it to satisfy $\varphi_0(x; a^{0}_{\eta(0)})$ it is enough to find some $\gamma \in \mathbb{Q} \subseteq \Gamma$ satisfying

$$\frac{c_{\eta(0)} - \sum_{\alpha=1}^{n} a^{\alpha}_{\eta(\alpha)}}{q} < \gamma < \frac{d_{\eta(0)} - \sum_{\alpha=1}^{n} a^{\alpha}_{\eta(\alpha)}}{q}.$$

□

There is a similar clasification of dp-finite (pure) fields, but lacking of explicit bounds of the dp-rank, due to Will Johnson. It is Theorem 1.3 in [11], and the definition below can be found in chapter 11 of [13].

Definition 3.1.7. Let $(K, v) \subseteq (L, w)$ be an extension of valued fields. We say that the extension is *defectless* if

$$[L : K] = [k_w : k_v][\Gamma_w : \Gamma_v],$$

and say that a henselian valued field (K, v) is *defectless* if its extension to the algebraic closure K^{alg} is defectless.

Theorem 11.27 of [13] asserts that any pseudo-complete valued field is defectless and, as we already know, henselian. Therefore the natural valuation of $K(\!(t^\Gamma)\!)$ is henselian and defectless for any field K and any ordered abelian group Γ.

Fact 3.1.8 (Cf. Theorem 1.3 of [11])**.** *A field K is dp-finite if and only if there is a henselian defectless valuation v on K satisfying the following conditions:*

- ✓ *The residue field k_v is elementary equivalent to $\mathbb{F}_p^{alg}$ or to some finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$.*
- ✓ *The value group Γ_v is dp-finite in $\mathscr{L}_{oag}$.*
- ✓ *If $char(K) = p > 0$, then Γ_v is p-divisible.*
- ✓ *If (K, v) is of mixed characteristic $(0, p)$, then $[-v(p), v(p)] \subseteq p\Gamma_v$.*

Moreover, Theorem 1.1 of [10] states that under the same conditions, if Γ_v has no singular primes, then (K, v) is *dp-minimal* as a valued field and therefore as a pure field whenever K is infinite. This can be stated as the following

Corollary 3.1.9. *If $char(K) = 0$ and K is a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$, and if Γ is an ordered abelian group without singular primes, then $K(\!(t^\Gamma)\!)$ is dp-minimal as a pure field. The same conclusion holds for $K = \mathbb{F}_p^{alg}$ and any p-divisible non-singular Γ.*

One natural question that arises from Fact 3.1.8 is if there are bounds for the dp-rank of such fields. If K has equicharacteristic zero and (K, v) admits an ac-map, Sinclair's Theorem 3.1.1 gives such bounds. Then we can ask how sharp this bound is, for example, for Hahn series fields $K(\!(t^\Gamma)\!)$.

As a first attempt, we may look at definability of v in $\mathscr{L}_{ring}$. A valued field (K, v) satisfying this condition would certainly satisfy that

$$\text{dp-rk}_{\mathscr{L}_{ring}}(K) = \text{dp-rk}_{\mathscr{L}_{div}}(K),$$

where $\mathscr{L}_{div}$ is defined as $\mathscr{L}_{ring}$ together with a binary relation symbol "$|$" to be interpreted as $x|y$ if and only if $v(x) \leq v(y)$, or equivalently, $yx^{-1} \in \mathcal{O}_v$. This is relevant because any inp-pattern in $\mathscr{L}_{oag}$ induces another inp-pattern in $\mathscr{L}_{div}$ of the same length. This is a consequence of the following

Proposition 3.1.10. *Let (K, v) be a valued field with value group Γ. Then, for any $\mathscr{L}_{oag}$-formula $\varphi(\overline{x})$ there exists some $\mathscr{L}_{div}$ formula $\varphi^*(\overline{x})$ such that for any $|\overline{x}|$-tuple $\overline{a} \in K$,*

$$\Gamma \models \varphi(v(\overline{a})) \iff (K, v) \models \varphi^*(\overline{a}).$$

Proof. We start by looking at $\mathscr{L}_{oag}$-terms. For constants in $\mathscr{L}_{oag}$, define $0^* = 1$ and $\infty^* = 0$. If $t_1(\overline{x}), t_2(\overline{x})$ are arbitrary $\mathscr{L}_{oag}$-terms, define $(t_1(\overline{x}) + t_2(\overline{x}))^* = t_1^*(\overline{x})t_2^*(\overline{x})$ and $(-t_1(\overline{x}))^* = t_1^*(\overline{x})^{-1}$. Then, for any $\overline{a} \in K$ and any $\mathscr{L}_{oag}$-term t, we have that

$$t(v(\overline{a})) = v(t^*(\overline{a})).$$

Now let $\varphi(\overline{x})$ be $t_1(\overline{x}) < t_2(\overline{x})$ or $t_1(\overline{x}) = t_2(\overline{x})$ for some $\mathscr{L}_{oag}$-terms t_1, t_2. Define $\varphi^*(\overline{x})$ to be $v(t_1^*(\overline{x})) < v(t_2^*(\overline{x}))$ and $v(t_1^*(\overline{x})) = v(t_2^*(\overline{x}))$ respectively. Because of the displayed equality, the conclusion holds for φ. After defining $(\neg\varphi(\overline{x}))^*$ and $(\varphi(\overline{x}) \wedge \psi(\overline{x}))^*$ to be $\neg\varphi^*(\overline{x})$ and $\varphi^*(\overline{x}) \wedge \psi^*(\overline{x})$ respectively, the inductive steps hold. Finally, if $\varphi(\overline{x})$ is the formula $\exists y \psi(y, \overline{x})$, then take $\varphi^*(\overline{x})$ as $\exists y \psi^*(y, \overline{x})$. If $\overline{a}$ is a $|\overline{x}|$-tuple of K, then

$$\begin{aligned} \Gamma \models \varphi(v(\overline{a})) &\iff \Gamma \models \psi(v(b), \overline{a}) \text{ for some } b \in K, \\ &\iff (K, v) \models \psi^*(b, \overline{a}) \\ &\iff (K, v) \models \varphi^*(\overline{a}). \end{aligned}$$

□

Therefore, if $d = \text{dp-rk}_{\mathscr{L}_{DP}}\left(K(\!(t^\Gamma)\!)\right)$, then

$$\text{dp-rk}_{\mathscr{L}_{oag}}(\Gamma) \leq \text{dp-rk}_{\mathscr{L}_{div}}\left(K(\!(t^\Gamma)\!)\right) \leq d.$$

The next result is now relevant. It is part 2 of Theorem 3.1 in [8].

Definition 3.1.11. An ordered abelian group Γ is said to be *p-regular* for a prime number p if and only if all proper quotients over convex subgroups of Γ are p-divisible. In addition, Γ is said to be *regular* if it is p-regular for all prime numbers p.

It is proved in [19] that regularly dense ordered abelian groups are elementary equivalent to archimedean groups. Since $\mathcal{G}_P$ is archimedean, we have trivially that it is regular.

Fact 3.1.12 (Cf. Theorem 3.1 of [8])**.** *Let (K, v) be a henselian valued field. If Γ_v is regular but not divisible, then v is definable without parameters, i.e. there is a parameter-free $\mathscr{L}_{ring}$-formula $\mu(x)$ such that $\mu(K) = \mathcal{O}_v$.*

If $\mathcal{O}_v$ is definable, then $\mathscr{M}_v$ is also definable, for any element of K is in $\mathscr{M}_v$ if and only if it is a non-invertible element of $\mathcal{O}_v$.

If the field $K(\!(t^\Gamma)\!)$ has dp-finite rank in $\mathscr{L}_{DP}$ then, as $K[\![t^\Gamma]\!]$ is definable in this language, it has also dp-finite rank in $\mathscr{L}_{DP}$ and therefore, as a pure ring, it is NIP. The following fact implies that this is all we need.

Fact 3.1.13 (Cf. Proposition 2.8 of [3]). *Let R be an integral domain and let S be a multiplicative subset of R.*

1. *If S is definable, then the burden of R equals the burden of $(S^{-1}R, R)$.*
2. *If S is externally definable in R and R is NIP, then $(S^{-1}R, R)$ is NIP and, by the latter item, $\text{dp-rk}(R) < \kappa \iff \text{dp-rk}(S^{-1}R, R) < \kappa$ for every cardinal κ.*

Note that whenever R is a valuation ring and $S = R\setminus\{0\}$, then $(S^{-1}R, R)$ is (interdefinable with) the valued field $K = \text{Quot}(R)$ with valuation ring R seen as an $\mathscr{L}_{div}$-structure. This is,

$$\text{dp-rk}_{\mathscr{L}_{ring}}(R) = \text{dp-rk}_{\mathscr{L}_{div}}(K, R)$$

whenever R is dependent.

Corollary 3.1.14. *Let K be a field of characteristic zero or $\mathbb{F}_p^{alg}$, and let Γ be an ordered abelian group. Let $d < \infty$ be the dp-rank of $K(\!(t^\Gamma)\!)$ in $\mathscr{L}_{DP}$. Then*

$$\text{dp-rk}_{\mathscr{L}_{oag}}(\Gamma) \leq \text{dp-rk}_{\mathscr{L}_{ring}}\left(K[\![t^\Gamma]\!]\right) \leq d.$$

Furthermore, if Γ is regular but not divisible, then

$$\text{dp-rk}_{\mathscr{L}_{oag}}(\Gamma) \leq \text{dp-rk}_{\mathscr{L}_{ring}}\left(K(\!(t^\Gamma)\!)\right) \leq d.$$

If we wanted to narrow the possible values of $\text{dp-rk}_{\mathscr{L}_{ring}}\left(K(\!(t^\Gamma)\!)\right)$, we could, for instance, declare K to be dp-minimal and Γ to be regular and not divisible, yielding that it is either $\text{dp-rk}_{\mathscr{L}_{oag}}(\Gamma)$ or $\text{dp-rk}_{\mathscr{L}_{oag}}(\Gamma)+1$. For example, take $\Gamma = \mathcal{G}_P$ for a fix set of prime numbers $P = \{p_1, \dots, p_n\}$, and take $K = \mathbb{C}, \mathbb{R}$ or $\mathbb{F}_q^{alg}$ for some prime number $q \notin P$.

Question 3.1.15. *What is the dp-rank of $K(\!(t^{\mathcal{G}_P})\!)$ and of $K(\!(t_1^\Gamma)\!)(\!(t_2^{\mathcal{G}_P})\!)$ as rings whenever $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_q^{alg}\}$ $(q \notin P)$ and $\Gamma \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{(p)}\}$?*

Definition 3.1.16. Let (K, v) be a valued field with value group Γ. A *cross section* is a group homomorphism $\iota : \Gamma \longrightarrow K^\times$ such that $v \circ \iota = id_\Gamma$.

Examples of cross sections are $\gamma \mapsto t^\gamma$ in $K(\!(t^\Gamma)\!)$ and $n \mapsto p^n$ in $\mathbb{Q}_p$. If (K, v) admits a cross section, then it admits an ac-map by defining it in x to be $\frac{x}{\iota(v(x))} + \mathscr{M}_v$ whenever $x \in K^\times$.

Cross sections are useful to translate inp-patterns from Γ_v to (K, v), but they are not everything we need to this end. Since Γ_v is ordered, we would like to be able to decide whether $\gamma_1 < \gamma_2$ by means of ι with a formula in $\mathscr{L}_{ring}$, which is, of course, equivalent to definability in $\mathscr{L}_{ring}$ of $\mathcal{O}_v$. Therefore, together with Fact 3.1.12 and Proposition 3.1.10, we can give an explicit inp-pattern in $K(\!(t^{\mathcal{G}_P})\!)$ of length $1 + |P|$.

Example 3.1.17. Since $\mathcal{G}_P$ is archimedean but not divisible, $K[\![t^{\mathcal{G}_P}]\!] = \mathcal{O}$ and $\mathscr{M}$ are $\mathscr{L}_{ring}$-definable in $K(\!(t^{\mathcal{G}_P})\!)$. Consider the following list of formulas and parameters in $K(\!(t^{\mathcal{G}_P})\!)$.

- ✓ Let $\varphi_0(x; y, z)$ be the formula $\frac{x}{y} \in \mathscr{M} \wedge \frac{z}{x} \in \mathscr{M}$. This formula says that $v(y) < v(x) < v(z)$. Take as parameters $(a_i^0)_{i<\omega} = (t^{c_i}, t^{d_i})_{i<\omega}$ for any sequence of tuples (c_i, d_i) in $\mathbb{Q}^\times$ such that $c_i < d_i < c_{i+1}$ for all $i < \omega$. It follows that $\{\varphi_0(x; a_i^0) : i < \omega\}$ is 2-inconsistent.

✓ For any $\alpha \in \{1, \dots, n\}$, let $\varphi_\alpha(x; y)$ be the formula $\exists \xi(\xi^{p_\alpha} y = x)$. Take as parameters $(a_i^\alpha) = (t^{b_i^\alpha})_{i<\omega}$, where $(b_i^\alpha)_{i<\omega}$ is the sequence of parameters of the α^{th} row of the inp-pattern in Proposition 3.1.6. Then $\{\varphi_\alpha(x, a_i^\alpha) : i < \omega\}$ is 2-inconsistent. Indeed, if $i \neq j$ and $f_1, f_2 \in K(\!(t^{\mathcal{G}_P})\!)$ are such that $f_1^{p_\alpha} t^{b_i^\alpha} = f_2^{p_\alpha} t^{b_j^\alpha}$, then $p_\alpha v(f_1) + b_i^\alpha = p_\alpha v(f_2) + b_j^\alpha$ and p_α divides $b_i^\alpha - b_j^\alpha$, which contradicts the choice of $(b_i^\alpha)_{i<\omega}$.

Let $\eta : \{0, \dots, n\} \longrightarrow \omega$ be any function. Then, following the argument of Proposition 3.1.6, the element t^{γ_η} satisfies $\{\varphi_0(x; a^0_{\eta(0)}), \dots, \varphi_n(x; a^n_{\eta(n)})\}$.

Note that we do not need the witness of $\{\varphi_0(x; a^0_{\eta(0)}), \dots, \varphi_n(x; a^n_{\eta(n)})\}$ to be exactly t^{γ_η}. Instead, we could use any element of the set $\{f \in K(\!(t^{\mathcal{G}_P})\!) : v(f) = \gamma_\eta\}$, which is known as the *fan* around γ_η. In particular, if we wanted to add another row to the latter inp-pattern, we could for instance take a formula $\psi(x; y)$ that divides in K (with parameters $(a_i)_{i<\omega}$) and declare the row to be $\{\psi(\mathrm{ac}(x), a_i) : i < \omega\}$ whenever ac is $\mathscr{L}_{ring}$-definable. This is not always the case: if K is algebraically closed and Γ is divisible, $K(\!(t^\Gamma)\!)$ is algebraically closed by Corollary 2.3.11, yielding that the set $\{f \in K(\!(t^\Gamma)\!) : \mathrm{ac}(f) = \mathrm{ac}(1)\}$ is either finite or cofinite. Since $K^\times$ is infinite and ac is surjective, this set must be finite, but it is not finite for it contains the ball $\{f \in K(\!(t^\Gamma)\!)^\times : v(f-1) > 0\}$, in light of Lemma 2.3.20. Since $\mathcal{G}_P$ is not divisible, we can hope that ac is, at least, $\mathscr{L}_{div}$-definable.

Question 3.1.18. *Is* ac *$\mathscr{L}_{ring}$-definable or $\mathscr{L}_{div}$-definable in $K(\!(t^{\mathcal{G}_P})\!)$?*

Note that if ac is $\mathscr{L}_{ring}$-definable in $K(\!(t^{\mathcal{G}_P})\!)$, we do not need to worry about adding another row to our inp-pattern, for as $\mathcal{O}$ is already $\mathscr{L}_{ring}$-definable, then $(K(\!(t^{\mathcal{G}_P})\!), v)$ as a ring has essentially the same structure as in $\mathscr{L}_{DP}$, and by Sinclair's Theorem 3.1.1 we would know exactly its dp-rank.

Another alternative in adding another row to our inp-pattern can be the following. Suppose $K[t^{\mathcal{G}_P}] := \{f \in K(\!(t^{\mathcal{G}_P})\!) : \mathrm{Supp}(f) \text{ is finite }\}$ is $\mathscr{L}_{ring}$-definable, and suppose moreover that it has a definable graduation: $K[t^{\mathcal{G}_P}] = \bigoplus_{n<\omega} G_n$ with $\mathscr{L}_{ring}$-definable $G_n := \{f \in K[t^{\mathcal{G}_P}] : |\mathrm{Supp}(f)| = n\}$. Then we could take the extra row to be $\{x \in G_i : i < \omega\}$. This is impossible because each of the G_n are infinite but they do not have interior (as in the $\mathscr{L}_{div}$-theory of Algebraically Closed Valued Fields, any infinite definable set must have interior):

Fact 3.1.19 (Cf. Lemma 2.3.3 of [17])**.** *Let X be a $\mathscr{L}_{DP}$-definable subset of a valued field (K, v) admitting* **VF***-quantifier elimination. The following statements are equivalent.*

1. *X has interior.*[8]
2. $\text{dp-rk}_{\mathscr{L}_{DP}}(X) = \text{dp-rk}_{\mathscr{L}_{DP}}(K)$.
3. $\text{dp-rk}_{\mathscr{L}_{DP}}(X) > 0$.

3.2 Ideals of Valuation Rings

Now we focus on the study of the poset of ideals of a valuation ring. Our interest in studying valuation rings in dp-finite fields comes from the idea of finding bounds to the dp-rank by means of the spectrum of the ring. In particular, the following facts aim to this direction.

[8]This is, X contains a ball $\{x \in K : v(x-c) > v(r)\}$ for some $c \in K$, $r \in K^\times$.

Fact 3.2.1 (Cf. Corollary 2.6 of [3]). *Let R be a domain of burden n. Then R has at most n maximal ideals.*

Fact 3.2.1 is a corollary of Prime Avoidance and the following

Proposition 3.2.2 (Cf. Proposition 2.3 of [3]). *Let R be a domain of burden n, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ be prime ideals of R. Then there is some $i \in \{1, \ldots, n+1\}$ such that $\mathfrak{p}_i \subseteq \bigcup_{j \neq i} \mathfrak{p}_j$.*

Proof. Suppose otherwise, so that for any $i \in \{1, \ldots, n+1\}$ there is some $a_i \in \mathfrak{p}_i$ satisfying $a_i \notin \bigcup_{j \neq i} \mathfrak{p}_j$. Since the ideals are prime, each of the sets $\mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$ is multiplicative, yielding that $a_i^k \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$ for any positive integer k. Define then the family of sets $X_i^k := (a_i^k) \setminus (a_i^{k+1})$, $i \in \{1, \ldots, n+1\}, k \in \mathbb{N}^*$. These sets are not empty because otherwise $(a_i^k) \subseteq (a_i^{k+1})$ implies that a_i is a unit, contradicting that $\mathfrak{p}_i$ is prime.

The family $\{X_i^k : i \in \{1, \ldots, n+1\}, k \in \mathbb{N}^*\}$ defines an inp-pattern of length $n+1$. Indeed, for any tuple of powers $(k_1, \ldots, k_{n+1})$ we have that

$$a_1^{k_1} \ldots a_{n+1}^{k_{n+1}} \in X_1^{k_1} \cap \ldots \cap X_{n+1}^{k_{n+1}}$$

because if $i \in \{1, \ldots, n+1\}$ and $a_1^{k_1} \ldots a_{n+1}^{k_{n+1}} = a_i^{k_i+1} b$ for some $b \in R$, then

$$a_1^{k_1} \ldots a_{i-1}^{k_{i-1}} a_{i+1}^{k_{i+1}} \ldots a_{n+1}^{k_{n+1}} \in (a_i) \subseteq \mathfrak{p}_i,$$

which implies that $a_j \in \mathfrak{p}_i$ for some $j \neq i$, a contradiction. We still need to prove that $\{X_i^k : k \in \mathbb{N}^*\}$ is 2-inconsistent for any $i \in \{1, \ldots, n+1\}$. Indeed, if $l < k$ and $r \in X_i^k \cap X_i^l$ and $r = ba_i^k$ for some $b \in R$, then $r = ba_i^{k-l-1} a_i^{l+1}$, which is absurd. $\square$

As a result of the latter proposition, if a dependent ring R has dp-rank n, then it does not admit any antichain of prime ideals of length $n+1$.

Proposition 3.2.3. *A ring R is a valuation ring if and only if its ideals are linearly ordered by inclusion. Hence any valuation ring is local.*

Proof. Recall that inverses must be taken within $\mathrm{Quot}(R)$. Let I_1, I_2 be ideals of R such that $I_1 \not\subseteq I_2$. Let $x \in I_1 \setminus I_2$, and let $y \in I_2$. In order to conclude that $y \in I_1$, we can argue that $yx^{-1} \in R$, for $y = yx^{-1}x$ and I_1 is an ideal. If that is not the case, then $xy^{-1} \in R$ and therefore $x = xy^{-1}y \in I_2$, which is absurd. This shows that the ideals of a valuation ring are linearly ordered by inclusion, hence R is local.

Moreover, suppose R is a ring whose ideals are linearly ordered by inclusion. Let $x = \frac{a}{b} \in \mathrm{Quot}(R)^\times$ be such that $x \notin R$. Then b does not divide a, i.e. $(a) \not\subseteq (b)$. Therefore $(b) \subseteq (a)$ and thus $x^{-1} = \frac{b}{a} \in R$, so R is a valuation ring. $\square$

More can be said about the ideals of a valuation ring $\mathcal{O}$. Lemma 2.3.1 of [5] states that if Γ is the value group associated to $\mathcal{O}$, then there is a one-to-one correspondence[9] between the set of prime ideals $\mathfrak{p}$ of $\mathcal{O}$ and the set of convex subgroups Δ of Γ given by

$$\begin{aligned} \mathfrak{p} &\mapsto \Delta_{\mathfrak{p}} = \{\gamma \in \Gamma : \gamma, -\gamma < v(x) \text{ for all } x \in \mathfrak{p}\}, \\ \Delta &\mapsto \mathfrak{p}_\Delta = \{x \in \mathcal{O} : v(x) > \gamma \text{ for all } \gamma \in \Delta\}. \end{aligned}$$

Therefore, if the rank of $\mathcal{O}$ is finite, then it coincides with its *Krull dimension* $\dim(\mathcal{O})$, which is defined as the maximal length of a chain of prime ideals. However, if there is some $\gamma \in \Gamma$ such that $[-\gamma, \gamma]$ is infinite, then $K[\![t^\Gamma]\!]$ is not Noetherian.

[9]This correspondence is in fact between ideals of $\mathcal{O}$ and *segments* of Γ : a segment is a subset Δ of Γ such that $\Delta = \cup_{\gamma \in \Delta}[-\gamma, \gamma]$. The content here is that *prime* ideals correspond to convex *subgroups.*

Observation 3.2.4. Let K be a dp-finite field and Γ a dp-finite ordered abelian group of rank k. Then $K[\![t^\Gamma]\!]$ has only one maximal ideal and has Krull dimension k. In particular, Corollary 3.1.14 implies that

$$\dim(K[\![t^{\mathcal{G}_{p,k}}]\!]) = k < 1 + k \leq \text{dp-rk}(K[\![t^{\mathcal{G}_{p,k}}]\!]).$$

However, if P is a finite set of prime numbers, it follows that

$$\dim(K[\![t^{\mathcal{G}_P}]\!]) = 1 < 1 + |P| \leq \text{dp-rk}(K[\![t^{\mathcal{G}_P}]\!]).$$

Question 3.2.5. *Are there valuation rings of arbitrary large Krull dimension but with fix dp-rank?*

If K is dp-minimal (say $\mathbb{R}$ or $\mathbb{F}_p^{alg}$) and Γ is dp-finite, then

$$\text{dp-rk}(\Gamma) \leq \text{dp-rk}(K[\![t^\Gamma]\!]) \leq \text{dp-rk}(\Gamma) + 1,$$

so we just have to find some $\Gamma' \equiv \Gamma$ with arbitrary large rank. The following definition and fact can be found in [15].

Definition 3.2.6. An ordered abelian group Γ is said to be *regularly dense* if for any positive integer n and any pair $\gamma_1 < \gamma_2$ of elements of Γ, there is some n-divisible $\gamma \in (\gamma_1, \gamma_2)$.

Fact 3.2.7 (Cf. Theorem 4.7 of [15])**.** *Two regularly dense ordered abelian groups Γ, Γ' are elementary equivalent if and only if $[\Gamma : p\Gamma] = [\Gamma' : p\Gamma']$ for any prime number p.*

Therefore, for any given n, $\mathbb{Q}^n$ ordered lexicographically is regularly dense and does not have singular primes, so $\mathbb{Q}^n \equiv \mathbb{Q}$. It follows from Fact 3.1.4 that any $\mathbb{Q}^n$ is dp-minimal, but has fix rank.

We can conclude, as a summary, that there is no reasonable relation between dp-rank and Krull dimension.

References

[1] H. Adler. Strong theories, burden, and weight. *Unpublished*, 2007. http://www.logic.univie.ac.at/~adler/docs/strong.pdf.

[2] L. Bélair. Types dans les corps valués munis d'applications coefficients. *Illinois J. Math.*, 43(2):410–425, 1999. https://archipel.uqam.ca/3522/1/belair_IJM1999.pdf.

[3] C. d'Elbée and Y. Halevi. Dp-minimal integral domains, 2020. https://arxiv.org/pdf/2006.05755.pdf.

[4] A. Dolich and J. Goodrick. A characterization of strongly dependent ordered Abelian groups. *Rev. Colombiana Mat.*, 52(2):139–159, 2018. http://www.scielo.org.co/pdf/rcm/v52n2/0034-7426-rcm-52-02-139.pdf.

[5] A. J. Engler and A. Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[6] R. Farré. Strong ordered abelian groups and dp-rank, 2017. https://arxiv.org/pdf/1706.05471.pdf.

[7] Y. Halevi and A. Hasson. Strongly dependent ordered abelian groups and henselian fields, 2018. https://arxiv.org/pdf/1706.03376.pdf.

[8] F. Jahnke and J. Koenigsmann. Defining coarsenings of valuations, 2015. https://arxiv.org/pdf/1501.04506v1.pdf.

[9] F. Jahnke, D. Palacín, and K. Tent, editors. *Lectures in model theory*. Münster Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2018. https://www.uni-muenster.de/imperia/md/content/logik/hils/mhnotes_v3.pdf.

[10] W. Johnson. On dp-finite fields, 2015. https://arxiv.org/pdf/1507.02745.pdf.

[11] W. Johnson. Dp-finite fields vi: the dp-finite Shelah conjecture, 2020. https://arxiv.org/pdf/2005.13989.pdf.

[12] I. Kaplansky. Maximal fields with valuations. *Duke Math. J.*, 9:303–321, 1942. http://projecteuclid.org/euclid.dmj/1077493226.

[13] F.-V. Kuhlmann. *Book on Valuation Theory*. In preparation, 2011. https://math.usask.ca/~fvk/Fvkbook.htm.

[14] A. MacIntyre. On ω_1-categorical theories of fields. *Fund. Math.*, 71(1):1–25, 1971.

[15] A. Robinson and E. Zakon. Elementary properties of ordered abelian groups. *Trans. Amer. Math. Soc.*, 96:222–236, 1960. https://www.ams.org/journals/tran/1960-096-02/S0002-9947-1960-0114855-0/S0002-9947-1960-0114855-0.pdf.

[16] P. Simon. *A guide to NIP theories*, volume 44 of *Lecture Notes in Logic*. Association for Symbolic Logic, Chicago, IL; Cambridge Scientific Publishers, Cambridge, 2015. https://doi.org/10.1017/CBO9781107415133.

[17] P. Sinclair. *Model Theory and Valuations*. PhD thesis, McMaster University, 2018. https://macsphere.mcmaster.ca/bitstream/11375/23326/2/Sinclair_Peter_TR_201808_PhD.pdf.

[18] L. van den Dries. Model theory of valued fields: Lecture notes. http://homepages.math.uic.edu/~freitag/valfields.pdf, 2004.

[19] E. Zakon. Generalized archimedean groups. *Trans. Amer. Math. Soc.*, 99:21–40, 1961. https://www.ams.org/journals/tran/1961-099-01/S0002-9947-1961-0120294-X/S0002-9947-1961-0120294-X.pdf.